Growth Options in General Equilibrium: Some Asset Pricing Implications[*]

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Abstract

We develop a general equilibrium model of a production economy which has a risky production technology as well as a growth option to expand the scale of the productive sector of the economy. We show that when confronted with growth options, the representative consumer may sharply alter consumption rates to improve the likelihood of investment. This reduction in consumption is accompanied by an erosion of the option value of waiting to invest, leading to investment near the zero NPV threshold. It also has important consequences for the evolution of risk aversion, asset prices and equilibrium interest rates which we characterize in this paper. One interesting prediction of the model is that we get time varying risk aversion and equity returns by virtue of the presence of growth option. We also find that the moneyness of the growth option is the key factor which determines the extent to which the book to market ratios will influence the conditional moments of equity returns.

Keywords: Time varying MRS; Growth Options; General Equilibrium.

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1   Introduction

This paper develops a framework to examine the exercise of growth options in general equilibrium. This framework is then used to derive implications regarding asset pricing quantities such as the equity risk premium. We construct a single good, general equilibrium model of an economy with a production technology that exhibits linear, stochastic constant returns to scale. This technology may be thought of as assets in place. The representative consumer can expand the scale of the production technology by investing an amount of the only good in the economy at any time. This is naturally interpreted as a growth option. We show that when confronted with growth option, the representative consumer may sharply alter consumption rates to improve the likelihood of investment. This reduction in consumption is accompanied by a fall in risk aversion in wealth and an erosion of the option value of waiting to invest. It also generates time-varying equilibrium interest rates and a countercyclical equity risk premium.

Equilibrium asset pricing in exchange economies with a representative consumer can be traced back to the papers of Lucas (1978) and Rubinstein (1976). Asset pricing models with production economies have been developed by a number of authors including Cox, Ingersoll and Ross (CIR, 1985), Kydland and Prescott (1982) and Jermann (1998). One of the difficulties with production economies is that the consumer is able to use the production technologies to smooth his consumption to a point that these models have difficulty in generating sufficient movements in the pricing kernel. Adjustment costs, time to build, and other enhancements have been explored by authors to overcome this difficulty. One way in which lumpy investments in production arises is through the growth options that economies often have. Recent examples of such growth options include the internet boom, fiber optic networks and communication technologies and bio-tech investments. Such growth options may not only significantly expand the scale of the economy but also the risk-return character of the economy. They require lump sum investments. This is the avenue that we pursue in this paper.

In the real options literature such lump sum investments in irreversible production technologies have been investigated [see Dixit and Pindyck (1994) for a survey of early real options models]. But these papers typically consider situations in which the exercise of real options does not affect equilibrium quantities such as consumption, asset prices, and interest rates. As a result, our model can be contrasted with the approaches

in this literature along the following dimensions:

1. In most real options models agents are risk neutral and maximize the discounted expected payoff of the real option. In our model, the decision maker is risk averse and maximizes the discounted expected life-time utility of consumption.

2. In option pricing theory, the dividend [consumption] policy is either exogenous or implicitly assumed away. In our framework the stream of dividends or consumption is *endogenously* determined. The investor must optimally select the sequence of consumption to implement an exercise policy that maximizes his life-time discounted expected utility. Because the optimal consumption and exercise policies depend on the scale of the project, the homogeneity property of standard option pricing models does not apply in our setup: doubling the strike price and the stock price does not double the option value.

3. In standard real options models, real options are redundant and do not affect the demand and supply of assets in the economy. In our framework, the presence of growth options has important consequences for asset pricing quantities such as equilibrium interest rates or the equity risk premium. One interesting prediction of the model is that we get time varying risk aversion and equity returns by virtue of the presence of growth option – the model produces a countercyclical risk aversion and a countercyclical equity risk premium. This comes from the production side of the economy as opposed to models in which habit formation produces time varying risk aversion to explain aggregate stock market behavior.

A number of authors have laid the foundations to link asset pricing with the presence of growth options in the economy. Notable contributions include Cochrane (1991, 1996), Berk, Green and Naik (1999), Kogan (2001), Wang (2001), Gomes, Kogan and Zhang (2003), and Cooper (2004).[1] Our focus as well as the modeling approach differ from these earlier contributions. While most of these papers examine the cross section of returns, our focus is on the equity risk premium. Moreover, we consider an

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[1] The paper that is most closely related to our analysis is Wang (2001). Wang builds a general equilibrium model where there are opportunities to adopt new technologies. While his paper provides some characteristics of the resulting equilibrium, the value-maximizing policies are not derived explicitly. Hence, it is difficult to gauge the effects of growth options on asset pricing quantities.

investment problem in which the economy has a simple (endogenous) structural break: The economy begins with a live growth option, but once it is optimally exercised it reverts to a "Robinson Crusoe" economy similar to CIR. One essential implication of this assumption is that we can get a closed-form solution for the optimization problem of the representative consumer. As a result, we can clearly distinguish our explanation for time-varying returns from competing explanations such as habit formation. In particular, our solution allows us to examine optimal exercise strategies in much greater detail and study how the feedback effects on equilibrium quantities influence optimal exercise. In addition, it allows us to explicitly link the role played by consumption in the exercise of growth options and hence on asset pricing.

At a basic level, the introduction of growth options is natural in production-based models of asset pricing. Economic intuition suggests that the consumer, when confronted with growth options, may sharply alter consumption rates to optimally exercise the growth option. This implies that the sensitivity of the consumption rate to wealth may be qualitatively different in economies with growth options than conventional models might suggest. This property will have a corresponding consequence for the evolution of risk aversion, asset prices and equilibrium interest rates which we characterize in this paper. In particular, because the value of the option depends on the endogenous time-varying consumption policy, the model generates a time-varying equity risk premium. We also find that significant fluctuations in moments of equity returns are predicted when the growth options are near the money. Specifically, the moneyness of the growth option is the key factor which determines the extent to which the book to market ratios will influence the conditional moments of equity returns.

The remainder of the paper is organized as follows. Section 2 reviews the baseline model of growth options. Section 3 develops the general equilibrium framework. Section 4 characterizes the endogenous level of the stock of good at which the option is exercised by the representative consumer. This trigger policy and its connection to optimal consumption are also addressed in section 4. Section 5 develops a complete characterization of asset prices in the economy. Section 6 concludes. Technical developments are gathered in the Appendix.

2 Benchmark model of real options

2.1 The basic setup

We begin by reviewing a standard model of real options in the spirit of McDonald and Siegel (1986). This will allow us to illustrate more clearly how the exercise of the growth option and asset pricing quantities interact in a general equilibrium framework.

2.1.1. Information structure.

We consider an infinite horizon production economy in continuous time. Uncertainty is represented by a probability space $(\Omega, \mathbb{F}, \mathcal{F}, P)$ on which is defined a standard Brownian motion Z. The filtration $\mathbb{F}$ is the usual augmentation of the filtration generated by the Brownian motion and we let $\mathcal{F} := \cup_{t \geq 0} \mathcal{F}_t$ so that the true state of nature is solely determined by the path of the Brownian motion. All processes are adapted to the filtration $\mathbb{F}$ and all statements involving random quantities hold either almost surely or almost everywhere depending on the context.

2.1.2. Production technology.

We consider a firm using a risky production technology that exhibits linear, stochastic constant returns to scale. At any time t, the firm's instantaneous cash flow is given by δW_t where δ is a strictly positive constant representing the rate at which dividends are paid and W is the non negative process defined by:[2]

$$dW_t = (\mu - \delta)W_t dt + \sigma W_t dZ_t, \qquad W_0 > 0, \tag{1}$$

where the instantaneous expected rate of return μ and volatility $\sigma > 0$ are exogenous constants. Below, we interpret W as the capital stock available to the firm.

2.1.3. Growth option.

At any time t, the stock of capital can be expanded by a factor $\alpha > 1$. Investment is irreversible and generates a net payoff $(\alpha - 1)W - I$, where $I > 0$ is a sunk cost that may be due to adjustment costs, learning by doing, or any other type of friction. Because the stock of capital is a sufficient statistic for the payoff from investment and this payoff is increasing in W, the optimal investment policy can be described by the first passage time of the process W to a constant threshold w_i.

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[2] We assume here an exogenous constant dividend yield. Later, in a general equilibrium setting, the consumer will choose optimally the consumption (dividend) policy.

2.2 Irreversibility, uncertainty, and investment

Firms in our economy own the production technology and one growth option to expand their stock of capital. As a result, firm value is equal to the value of assets in place plus the value of the growth option. Assume that the investor is risk neutral and denote his discount rate by ρ. The value of assets in place is then given by:

$$E\left[\left.\int_t^\infty e^{-\rho(s-t)}\delta W_s ds\right| \mathcal{F}_t\right] = \frac{\delta}{\rho - \mu + \delta}W_t = \Pi W_t. \tag{2}$$

Consider next the value of the growth option. At the time τ of investment the stock of capital is increased by a factor α. As a result, the payoff from exercising the growth option is $G(W_\tau) := (\alpha - 1)\Pi W_\tau - I$ and we can write firm value as:

$$V(W_0) = \Pi W_0 + \sup_{\tau \in \mathcal{S}} E\left[e^{-\rho\tau}G(W_\tau)1_{\{\tau < \infty\}}\right], \tag{3}$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}-$stopping times, τ is the random time of investment, and the second term on the right hand side accounts for the value of the growth option. Standard calculations give:

$$V(W_0) \equiv \Pi W_0 + [(\alpha - 1)\Pi w_i - I]\left(\frac{w_i}{W_0}\right)^{-\beta}, \qquad W_0 < w, \tag{4}$$

where the selected investment threshold w_i satisfies:

$$\left(\frac{\alpha - 1}{I}\right)\Pi w_i = \frac{\beta}{\beta - 1} > 1, \tag{5}$$

and $\beta > 1$ is defined by:

$$\beta = \frac{1}{\sigma^2}\left[m + \sqrt{m^2 + 2r\sigma^2}\right], \qquad \text{for} \quad m = \frac{\sigma^2}{2} - (\mu - \delta). \tag{6}$$

Equation (4) shows that firm value equals the value of assets in place plus the value of the growth option. The value of the growth option equals the product of the investment surplus $[(\alpha - 1)\Pi w_i - I]$ and a stochastic discount factor $[(w_i/W)^{-\beta}]$. This discount factor accounts for both the timing of investment and the probability of investment, and captures the state contingent nature of the model. Equation (5) gives the optimal investment rule, that is the critical value w_i at which it will be optimal to invest. Because $\beta > 1$, we have $(\alpha - 1)\Pi w_i > I$. Thus, irreversibility and the ability

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to delay lead to a range of inaction even when the net present value of the project is positive. The relative size of this region of inaction only depends on $\frac{\beta}{\beta-1}$: whether the option doubles or triples the capital stock, the ratio $\left(\frac{\alpha-1}{I}\right)\Pi w_i$ will still be the same. This homogeneity property, or scale independence, of the optimal investment policy is well known in the literature, see Hubbard (1994).

As shown by equation (5), the investment threshold depends on the parameters of the production technology, the subjective discount rate ρ, and the payout rate δ. Notably, w_i increases with σ and decreases with ρ and δ. In this benchmark model, any dividend payout is *exogenous.* Hence, the investment threshold and the value of the firm only depend on the dividend payout policy through the exogenous payout rate δ. When the subjective discount rate is high enough, the consumer prefers a positive dividend payout. Dividends increase the value derived from current consumption, but reduce the value of the growth option. This tension, *in an equilibrium model,* determines the optimal payout policy *endogenously.*

We have so far sketched out the trade-offs between current consumption and the exercise of growth options without enforcing any equilibrium requirements. In the rest of the paper, we focus on a general equilibrium setting where the representative consumer chooses both the payout policy and the exercise time of the growth option so as to maximize his lifetime expected utility of consumption. The focus is then on the manner in which these decisions may influence equilibrium quantities.

3   The equilibrium model

In this section we extend the analysis to consider a single good production economy with a representative consumer. To do so, we first present the assumptions underlying the equilibrium model. We then review the solution provided by Cox, Ingersoll and Ross (1985) to the consumer's optimization problem in the absence of growth options.

3.1   The economy

3.1.1. Investment opportunities.

There is a single consumption good which is taken as a numéraire and we assume that there are two investment opportunities available to the representative consumer.

The first investment opportunity is a locally risk-free bond which pays no dividends. We denote its price process by B and assume that

$$dB_t = B_t \left(r_t dt + dL_t \right), \qquad B_0 = 1$$

for some instantaneous rate of return r and some singularly continuous process L which are to be determined endogenously in equilibrium. As in other equilibrium models of investment, see for example Casassus, Collin-Dufresne and Routledge (2004), the presence of a singular component in the equilibrium price is necessary to account for the singularity induced by the exercise of the growth option.

The second investment opportunity is a risky production technology whose output is the consumption good. As before, this technology exhibits linear returns to scale and the dynamics of its instantaneous return are given by

$$dS_t = S_t \left(\mu dt + \sigma dZ_t \right), \qquad S_0 > 0. \tag{7}$$

where the instantaneous expected rate of return μ and instantaneous volatility $\sigma > 0$ are exogenous constants. In addition to these two investment opportunities, we assume that the representative consumer can invest I units of the consumption good to expand the scale of the production technology by a constant factor $\alpha > 1$.

3.1.2. Consumption and investment plans.

Trading takes place continuously and there are no market frictions. A consumption and investment plan is a triple (c, π, τ) where c is a non negative process representing the agent's rate of consumption, π is a process representing the amount of capital invested in the production technology and τ is a stopping time representing the exercise time of the growth option.

Assume that the consumer follows an arbitrary consumption and investment plan and denote by W the corresponding wealth process. Since the consumer's actions must be self financing, $W - \pi$ is invested in the risk-free security. Moreover, the exercise of the growth option at the stopping time τ induces the consumer's wealth process to jump by the amount

$$\Delta(\pi_\tau) := \lim_{\delta \to 0} \left[W_{\tau + |\delta|} - W_\tau \right] = (\alpha - 1)\pi_\tau - I.$$

As a result, the dynamics of the consumer's wealth are given by

$$W_t = W_0 + \int_0^t (W_s - \pi_s) \frac{dB_s}{B_s} + \int_0^t \pi_s \frac{dS_s}{S_s} - \int_0^t c_s ds + 1_{\{t > \tau\}} \Delta(\pi_\tau). \qquad (8)$$

Throughout the paper, we say that a consumption and exercise plan (c, τ) is *feasible* if there exists π such that the corresponding solution to the above equation is well defined and non negative. As shown by Dybvig and Huang (1989), this restriction is sufficient to rule out arbitrage opportunities from the market.

3.1.3. Preferences, endowments and equilibrium.

The agent is endowed with an initial wealth $W_0 > 0$ and has *exclusive access* to the risky production technology and the ensuing growth option. His preferences over consumption plans are represented by the lifetime expected utility functional

$$E\left[\int_0^\infty e^{-\rho t} U\left(c_t\right) dt\right] = E\left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt\right], \qquad (9)$$

where $1 \neq R > 0$ is the consumer's constant relative risk aversion and $\rho \geq 0$ is his subjective rate of time preference. While most of our results hold for general utility functions satisfying the Inada conditions, we focus on this simple specification of the model because it allows us to obtain explicit results in some cases of interest and has been widely used in asset pricing.

An equilibrium for our production economy with growth option is reached if the bond price process is such that the representative consumer optimally invests all of his wealth into the production technology and nothing in the riskless asset. More formally, we will use the following definition.

Definition 1 *An equilibrium for the above economy is a stopping time $\tau^* \in \mathcal{S}$ and a collection of stochastic processes (B, W, c^*, π^*) such that*

a) *Given the bond price process, the pair (c^*, τ^*) maximizes the lifetime utility of the representative consumer over the set of feasible plans and is financed by π^*.*

b) *The process W solves equation (8) given (c^*, π^*, τ^*), the bond price process and the initial stock of goods in the economy.*

c) *The markets for the riskless asset and the production technology clear in the sense that $\pi^* = W$ holds at all times.*

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3.2 The benchmark equilibrium: CIR (1985)

Set $\alpha = 1$ and $I = 0$ so that the consumer's wealth becomes independent of the growth option. In this case, our economy collapses to a simple version of the Cox, Ingersoll and Ross (1985) economy. For the purpose of comparison, and in order to facilitate the analysis of later sections, we now briefly review some of their results.

Under the assumptions about preferences and technology set forth above, a unique equilibrium exists in this production economy without growth options if and only if the *marginal propensity to consume from wealth*

$$\Lambda := \frac{\rho}{R} + \frac{1-R}{R}\left(\frac{\sigma^2 R}{2} - \mu\right) \tag{10}$$

is strictly positive. Under this parametric restriction, which merely requires the rate of time preference to be high enough, the bond price process is continuous, the equilibrium interest rate is

$$r_t^0 = r^0 := \mu - \sigma^2 R, \tag{11}$$

and the representative consumer's optimal consumption policy is given by

$$c_t^0 = c^0(W_t) := \Lambda W_t. \tag{12}$$

The lifetime utility of the representative consumer can also be computed explicitly. Specifically, plugging the optimal consumption policy defined above into equation (9) and computing the conditional expectation we find

$$V_0(W_0) := U'(\Lambda)U(W_0). \tag{13}$$

When there are many risky technologies, the CIR (1985) model gives very similar conclusions: The risk-free rate is still a constant and the functional forms of the optimal consumption and the value function are still the same with the exception that the drift and the volatility parameters will now reflect the weighted drifts and weighted covariances of the outputs from the different technologies.

The above equilibrium not only serves as a benchmark for our economy with growth options, it also gives the boundary condition for computing the equilibrium of our model. Indeed, once the growth option is exercised our economy again collapses to that of Cox, Ingersoll and Ross (1985) and hence admits a unique equilibrium similar to the one described above.

4 The central planner's problem

This section formulates the central planner's problem, that is the choice of an optimal consumption plan and an optimal investment time given that the market clears.

4.1 Formulation

In equilibrium, the representative consumer optimally invests all his wealth in the production technology. His dynamic budget constraint can thus be written as

$$W_t = W_0 + \int_0^t W_s \left(\mu ds + \sigma dZ_s \right) - \int_0^t c_s ds + 1_{\{t>\tau\}} \Delta(W_\tau). \tag{14}$$

The central planner's problem consists in maximizing the lifetime utility subject to the non negativity constraint and the above dynamic budget constraint. In order to facilitate the presentation of our result, let Θ denote the set of admissible plans, that is the set of consumption and investment plans (c, τ) such that

$$E \left[\int_0^\infty e^{-\rho s} |U(c_s)| ds \right] < \infty$$

and the corresponding solution to (14) is non negative throughout the infinite horizon. With this notation, the value function of the central planner is

$$V(W_0) := \sup_{(c,\tau) \in \Theta} E \left[\int_0^\infty e^{-\rho t} U(c_t) dt \right].$$

The following lemma provides a necessary and sufficient condition on the parameters of the model for this problem to be well defined.

Lemma 1 *The central planner's value function is finite if and only the marginal propensity to consume from wealth Λ is strictly positive.*

4.2 The optimal policy

Once the growth option is exercised, the central planner optimally follows the consumption plan prescribed by equation (12). Hence, his value function at the exercise time of the growth option is given by

$$V_1(W_\tau) := V_0 \left(\alpha W_\tau - I \right),$$

where V_0 is the value function absent growth option defined in (13). An application of the dynamic programming principle, which is justified in Appendix A, then yields

$$V(W_0) = \sup_{(c,\tau) \in \Theta} E \left[e^{-\rho\tau} V_1(W_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-\rho t} U(c_t) dt \right], \tag{15}$$

and it follows that we can from now on focus on the determination of the optimal exercise time and the optimal consumption plan prior to exercise.

In order to provide the intuition behind our main results, we start by providing a heuristic description of the solution. Since the representative consumer can always decide to postpone indefinitely the exercise of the growth option, we have

$$V_0(w) \leq V(w), \qquad w \in (0, \infty),$$

where V_0 is defined in (13). In order to provide a differential characterization of the value function, consider first that it is strictly suboptimal to exercise the growth option over the interval $[t, t + \varepsilon]$. In this situation, it follows from the definition of the value function and the dynamic programming principle that

$$V_1(W_t) < V(W_t) = \sup_{c \geq 0} E \left[e^{-\rho\delta} V(W_{t+\delta}) + \int_t^{t+\delta} e^{-\rho(s-t)} U(c_s) ds \, \middle| \, \mathcal{F}_t \right]$$

where $W_{t+\delta}$ is the central planner's wealth at time $t + \delta$ determined according to the stochastic differential equation (14). In differential form, this implies that the value function must satisfy the Hamilton-Jacobi-Bellman equation $\mathcal{D}V = 0$ where the second order differential operator $\mathcal{D}$ is defined by

$$\mathcal{D}V := \inf_{c \geq 0} \left\{ \rho V - \mu w V' - \frac{1}{2} \sigma^2 w^2 V'' - U(c) + c V' \right\}. \tag{16}$$

In accordance with the literature on optimal stopping problems, we refer to the set of points $w \in (0, \infty)$ such that $V(w) > V_1(w)$ as the *continuation* region.

Consider next a situation where the central planner's wealth at time t is such that it is optimal to exercise the growth option immediately. In this case,

$$V_1(W_t) \geq V(W_t) > \sup_{c \geq 0} E \left[e^{-\rho\delta} V(W_{t+\delta}) + \int_t^{t+\delta} e^{-\rho(s-t)} U(c_s) ds \, \middle| \, \mathcal{F}_t \right].$$

The first inequality states that immediate investment is optimal. Furthermore, under the optimal policy strict inequality is impossible since the central planner is free to

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exercise the option instantaneously. Thus, the first inequality is always satisfied as an equality and $V(W_t) = V_1(W_t)$ in the *exercise* region. The second inequality captures the fact that the central planner can always postpone investment, but currently finds this to be strictly suboptimal. In differential form, this second inequality implies that we must have $\mathcal{D}V > 0$ in the exercise region.

Putting together the different conditions stated above, we obtain a single variational inequality to be satisfied by the value function

$$\min\{V(w) - V_1(w), V(w) - V_0(w), \mathcal{D}V(w)\} = 0, \qquad w \in (0, \infty). \qquad (17)$$

This equation shows that in our model the central planner's value function must lie above the obstacle

$$V_*(w) := \max\{V_0(w), V_1(w)\} = \begin{cases} V_0(w), & w \leq \frac{I}{\alpha - 1} \\ V_1(w), & \text{otherwise,} \end{cases}$$

and that the state space of the problem can be divided into three regions. Specifically, we expect that there exists two constant wealth levels w_c and w_i such that:

1. If the central planner's wealth belongs to the set $\mathcal{B}_0 := (0, w_c]$ then it is suboptimal to curtail current consumption in order the favor the exercise of the growth option and future consumption. As a result, the optimal consumption plan is not affected by the growth option in the sense that

$$c(W_t) = c^0(W_t) = (V_0'(W_t))^{-\frac{1}{R}} = \Lambda W_t. \qquad (18)$$

   Furthermore, it is not optimal to exercise the growth option and the value function of the representative consumer satisfies $V = V_0$.

2. If the central planner's wealth belongs to $\mathcal{C} := (w_c, w_i)$, then it is optimal to reduce current consumption in order to increase the likelihood of exercise. Over this region, the value function lies strictly above the obstacle $V_*(w)$ and satisfies the HJB equation $\mathcal{D}V = 0$. Furthermore, the optimal consumption rate is given by

$$c(W_t) = (V'(W_t))^{-\frac{1}{R}}$$

   and the exercise of the growth option occurs as soon as the central planner's wealth reaches the constant investment threshold w_i.

3. If the central planner's wealth belongs to $\mathcal{B}_1 := [w_i, \infty)$, then it is optimal to exercise the growth option immediately. In this case, the value function satisfies $V = V_1$ and the optimal consumption policy after investment is given by (18).

The continuation region of the problem is thus given by $\mathcal{B}_0 \cup \mathcal{C} = (0, w_i)$ while the exercise region is given by $\mathcal{B}_1 = (w_i, \infty)$. Finally, we expect the value function to be smooth almost everywhere except possibly at the investment threshold and to satisfy the smooth pasting condition

$$\lim_{w \uparrow w_i} V'(w) = \lim_{w \downarrow w_i} V_1'(w) = \alpha \left[\Lambda(\alpha w - I) \right]^{-R},$$

on the common boundary of the continuation and exercise regions. The decomposition of the state space described above and the expected shape of the central planner's value function are illustrated in the following figure.

[Insert Figure 1 Here]

When solving the optimization problem of the representative consumer, it is important to note that the differential part of the variational inequality is non linear due to the presence of the term

$$U^*(V'(w)) := \sup_{c \geq 0} \left\{ U(c) - cV'(w) \right\}$$

As a result it is by no means trivial to verify that the value function is smooth enough to satisfy (17) in the classical sense and standard existence theorems for classical solutions of obstacle problems cannot be applied to our case. In order to circumvent this difficulty we use an approach which was suggested by Shreve and Soner (1994). In a first step, we show that the central planner's value function is concave and solves (17) in the viscosity sense. Relying on these properties, we prove in a second step that the value function is a classical, i.e. almost everywhere twice continuously differentiable, solution to the obstacle problem. Finally, we establish in a third step the different properties of the optimal strategy discussed above. We report the details in Appendix B.

The following theorem provides a complete solution to the central planner's problem and constitutes our main result.

Theorem 1 *Assume that the condition of Lemma 1 hold. Then the central planner's value function is the unique classical solution to the obstacle problem (17) such that*

$$V(w) \leq V_0(\alpha w), \qquad w \in (0, \infty).$$

Furthermore, the central planner's value function is once continuously differentiable everywhere and there exist two non negative constant w_c, w_i with

$$0 \leq w_c < w_e := \frac{I}{\alpha - 1} < w_i < \infty,$$

such that the central planner's value function is twice continuously differentiable everywhere except at the point w_i and

$$\mathcal{B}_0 := \{w \in (0, \infty) : V(w) = V_0(w)\} = (0, w_c],$$
$$\mathcal{C} := \{w \in (0, \infty) : V(w) > V_*(w)\} = (w_c, w_i),$$
$$\mathcal{B}_1 := \{w \in (0, \infty) : V(w) = V_1(w)\} = [w_i, \infty).$$

Finally, the optimal exercise time and the central planner's optimal consumption policy prior to investment are explicitly given by

$$\tau^* := \inf\{t \geq 0 : W_t^* \in \mathcal{B}_1\} = \inf\{t \geq 0 : W_t^* \geq w_i\},$$
$$c_t^* := 1_{\{W_t^* \in \mathcal{B}_0\}} c^0(W_t^*) + 1_{\{W_t \in \mathcal{C}\}} \left(V'(W_t^*)\right)^{-\frac{1}{R}},$$

where the non negative process W^* is the unique solution to the stochastic differential equation (14) associated with the consumption and exercise plan (c^*, τ^*).

Proof. See Appendix B.∎

Theorem 1, which completely characterizes the central planner's problem leads to several insights. First, the value function of the central planner, in general, has three regions of economic interest. The first region, corresponding to very low wealth levels, is one in which the growth options do not add to the welfare of the central planner. The economy is "too poor" to benefit from the existence of growth options. The second region, where the economy has reached a wealth level that is high enough that the presence of growth options begins to influence optimal consumption levels, but not high enough for the economy to exercise the options. This is perhaps the economically the

most interesting region in our model. Finally, when the wealth level is sufficiently high, the growth option is exercised, and the economy reverts back to the Cox, Ingersoll and Ross (1985) economy. In addition, theorem 1 shows the manner in which the structure of optimal consumption policy is influenced by the presence of growth options: the current level of wealth as well as the critical level of wealth at which the option will be exercised play a role in the optimal consumption policy, in general.

When the rate of time preference of the representative consumer is equal to zero, his optimization problem admits a closed-form solution. According to Lemma 1, this undiscounted optimization problem imposes stronger restrictions on the parameters of the model. However, it also allows us to develop the intuition underlying the solution to the general optimization problem of the consumer. In the following, we first present the complete solution to the undiscounted case. We then provide a numerical solution to the discounted case that builds on the insights of the closed-form solution.

4.3   The undiscounted case

In this section, we provide a complete solution to the undiscounted optimization problem of the representative consumer. As shown in Lemma 1, the value function of the consumer is finite if and only the marginal propensity to consume from wealth Λ is strictly positive. Moreover, the restrictions imposed by Lemma 1 on the parameters of the production technology depend on whether the coefficient of relative risk aversion is larger or smaller than one. As we show below, the value of the risk aversion coefficient is also essential in determining the optimal exercise time and the central planner's optimal consumption policy.

Proposition 1 *Assume that the problem is undiscounted, that $R < 1$ and that the conditions of Lemma 1 hold true. Define a non negative function by setting*

$$\Phi(x,w) := \left\{ V_0'(x)^{\frac{1}{R}} + \left(\frac{w}{x}\right)^{\frac{2\mu}{\sigma^2 R}} \left[V_1'(w)^{\frac{1}{R}} - V_0'(w)^{\frac{1}{R}} \right] \right\}^R.$$

Then we have $w_c = 0$ and the optimal investment threshold is given by the unique solution in the interval (w_e, ∞) to the non linear equation

$$V_1(w_i) - \int_0^{w_i} \Phi(x, w_i) dx = 0.$$

Furthermore, the central planner's value function and optimal consumption strategy prior to investment are given by

$$V(w) = V_1(w_i \vee w) - \int_{w \wedge w_i}^{w_i} \Phi(x, w_i) dx,$$

and $c_t^* = V'(W_t^*)^{-\frac{1}{R}}$ where the non negative process W^* is the unique solution to the stochastic differential equation (14) associated with the pair (c^*, τ^*).

Proposition 1 shows that with zero discounting and $R < 1$, the value function has only two regions of economic interest: growth options always matter in this special case. This is intuitive, as the consumer's subjective rate of discount is zero, future consumption possibilities are given as much weight as current consumption possibilities. In addition, the proposition makes it clear that the optimal investment threshold is influenced by the parameters of the technology in place (see equation (7)), the risk aversion coefficient, as well as the current wealth level. When the coefficient of relative risk aversion of the representative consumer is larger than one, the investment threshold and the value function of the consumer admit simple expressions. In addition, the central planner's optimal consumption policy is not affected by the presence of the growth option. These results are illustrated by Proposition 2 below.

Proposition 2 *Assume that the problem is undiscounted, that $R > 1$ and that the conditions of Lemma 1 hold. Then the investment threshold is given by*

$$w_i := w_i(R) = \frac{I}{\alpha - \alpha^{\frac{1}{R}}} \geq \lim_{R \to \infty} w_i(R) = w_e,$$

and we have $w_c = 0$. Furthermore, the central planner's value function and optimal consumption strategy prior to investment are given by

$$V(w) = V_1(w_i \vee w) - V_0(w_i \vee w) + V_0(w),$$

and $c_t^* = c^0(W_t^*) = \Lambda W_t^*$ where the non negative process W^* is the unique solution to the stochastic differential equation (14) associated with the pair (c^*, τ^*).

Proposition 2 provides the optimal investment and consumption rules for the representative consumer when there is no discounting and $R > 1$. As shown in the Proposition, the optimal investment threshold is independent of the parameters of the stochastic process that underlie the productive technology. This is a surprising result: the

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central planner determines the exercise policy solely on the basis of information about the growth option and risk aversion. The central planner ignores information about the drift rate and the volatility of the assets in place. For *large values of risk aversion*, the optimal investment threshold rapidly decreases and *the traditional NPV rule becomes increasingly descriptive of the optimal investment policy* (see the discussion on investment policy below).

In the analysis below, we focus on the case where $R < 1$ to develop the intuition to the general solution with discounting. Indeed, this case is particularly interesting since the elasticity of intertemporal substitution is less than 1. As a result, the representative consumer is willing to reduce his current consumption to improve his future consumption stream. In the discounted case, such a behavior also arises when $R > 1$.

4.3.1. Investment policy.

Consider first the selected investment policy. As a in standard real options models, the selected investment threshold w_i depends on the parameters of the production technology and the cost of investment I. However, in our setup it also reflects the attitude of the representative consumer towards risk. Figure 2 plots the investment threshold w_i as a function of the relative risk aversion coefficient, R.

[Insert Figure 2 Here]

As shown in the figure, the investment threshold decreases with risk aversion (the first order derivative of w_i with respect to R in Proposition 2 shows that this is also the case when $R > 1$). To understand this feature of the model, note that before investment, the representative consumer owns the production technology and a growth option to expand the stock of capital. By investing, the consumer transforms an asset with volatility $\Theta\sigma$ into an asset with volatility σ, where Θ is the elasticity of the growth option with respect to wealth (see section 5.2 below). Because this elasticity is greater than one, investing in the asset reduces risk and thus improves the consumer's utility. This in turn provides the representative consumer with an additional benefit to investment. Because the improvement in utility increases with risk aversion, the selected investment threshold decreases with risk aversion. When $R > 1$, the investment threshold converges to the 0-NPV rule as risk aversion increases. That is, risk aversion

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erodes the option value of waiting to invest. *The concavity of the value function contributes to the fact that the scale of the growth option is a relevant factor in the exercise decision* (i.e. w_i depends on I even though we hold $(\alpha - 1)/I$ fixed).

4.3.2. Optimal consumption.

In standard real options models, the investor has no choice over the consumption stream over his life. Thus, no matter how attractive the growth option is, the investor still consumes at the same rate δ prior to the exercise of the growth option. On the contrary, our equilibrium model allows the consumer to choose the consumption sequence simultaneously with the optimal exercise strategy. In the equilibrium model without growth options, the optimal consumption rate is given by $c^0(W_t^*) = \Lambda W_t^*$. Thus, the elasticity of consumption with respect to wealth is constant. In the presence of a growth option, we have

$$c\left(W_t^*\right) = \left\{ V_0'\left(W_t^*\right)^{\frac{1}{R}} + \left(\frac{w_i}{W_t^*}\right)^{\frac{2\mu}{\sigma^2 R}} \left[V_1'\left(w_i\right)^{\frac{1}{R}} - V_0'\left(w_i\right)^{\frac{1}{R}}\right] \right\}^{-1} < \Lambda W_t^*, \qquad 2\mu < \sigma^2 R.$$

This expression reveals that when the growth option is very attractive, the consumer can reduce his rate of consumption to a level that is less than the long-run equilibrium value to accelerate the exercise. This effect is described in Figure 3 below.

[Insert Figure 3 Here]

Figure 3 shows that the consumption rate relative to the long-run value dramatically declines as the growth option goes into the money and jumps to its CIR level at when the capital stock reaches w_i. A simple examination of the dynamic budget constraint before investment reveals that the consumer reduces his consumption to accelerate the exercise of the growth option. In particular, for any investment threshold w_i, the expected time to investment decreases as consumption decreases.

4.4   The discounted case.

When the rate of time preference is strictly positive ($\rho > 0$) the optimization problem of the representative consumer cannot be solved explicitly. The approach to solve this problem is again by backward induction. We first solve the general equilibrium

of the economy *after* the growth option is exercised. This follows directly from the contributions of Cox, Ingersoll and Ross (1985). We then use this value function as the boundary condition to search for the optimal exercise boundary of the consumer. The details are reported in the Appendix.

4.4.1. Investment policy.

As in the undiscounted case, we first analyze the selected investment policy by examining a baseline case in which the expected growth rate of the capital stock ignoring consumption is $\mu = 0.15$. In Table 1 below, we report the investment threshold w_i assuming that the ratio $(\alpha - 1)/I$ is fixed at 1. The long-run marginal propensity to consume (MPC) out of wealth is shown in parentheses. This MPC is computed using the CIR equilibrium that prevails in our economy after the growth option is exercised. Note that as the subjective rate of discount increases, the marginal propensity to consume increases, increasing the incentive to exercise early. Thus, the growth option is exercised sooner. The table also shows that the selected investment threshold depends on I, even though we hold the ratio $(\alpha - 1)/I$ constant. *Thus, the scale of the growth option matters in general equilibrium.*

[Insert Table 1 Here]

4.4.2. The erosion of the option value of waiting to invest.

One of the major contributions of the real options literature is to show that with uncertainty and irreversibility, there exists a value of waiting to invest and the decision maker should only invest when the asset value exceeds the investment cost by a potentially large option premium. This effect is well summarized in the survey by Dixit and Pindyck (1994). These authors write:

> "We find that for plausible ranges of parameters, the option value [of waiting] is quantitatively very important. Waiting remains optimal even though the expected rate of return on immediate investment is substantially above the interest rate or the normal rate of return on capital. Return multiples as much as two or three times the normal rate are typically needed before the firm will exercise its option and make the investment."

In order to compare our results with the baseline model of real options, it is essential to pin down dividend yields that are specified exogenously and adjust for the presence of risk premium. We will set the exogenous dividend yield at the long-run marginal propensity to consume (MPC) implied by the CIR (1985) model given the parameters that we use in the baseline model. The issue of risk premium is fully addressed in the general equilibrium model developed in the paper. We provide a comparison of the optimal exercise policies with the baseline case below in Table 2. In this table, we report the wedge of inaction $\beta/(\beta - 1)$ when the drift rate is $\mu = 15\%$. We report in parentheses, the long run MPC implied by the CIR model.

[Insert Table 2 Here]

Table 2 shows that risk aversion decreases the investment threshold selected by the manager, and hence speeds up investment. One direct implication of this result is that risk aversion significantly erodes the value of waiting to invest. For example, when $\rho = 12\%$, and $\sigma = 0.20$, the exercise occurs in the baseline model at 1.91 as opposed to a wealth level of 1.28 in Table 1. In this example the 0–NPV investment threshold is 1. Thus the equilibrium model predicts that investment should occur when the present value of cash flows exceed the cost of investment by 28% whereas the baseline model predicts that it should occur when the present value of cash flows exceed the cost of investment by 91%. Our analysis shows that the feedback effects that growth options have on consumption may be rather important when viewed in the context of an aggregate economy.

4.4.3. Optimal consumption.

For completeness, Figure 4 plots the optimal normalized consumption $(\Lambda W_t)^{-1}c(W_t)$ as a function of the initial wealth of the representative consumer. As in the case without discounting, the consumption rate relative to the long-run value dramatically declines as the growth option goes into the money. Again, as the scale of the project goes down the effect on consumption rates becomes predictably much less pronounced.

[Insert Figure 4 Here]

5 Asset pricing implications

5.1 Time-varying risk aversion and interest rates

The solution to the central planner's problem can be used to construct a competitive equilibrium in a decentralized production economy [see e.g. Kogan (2001) for the construction of such an equilibrium]. The pricing kernel or state price density for this economy is given by $\xi_t = e^{-\rho t} U'(c_t)$. Because the state price density is determined by the consumption of the representative consumer, it exhibits a singularity at the time of investment. Specifically, at the time τ of investment, the consumption rate goes back to its long-run equilibrium value $c^0(W_t^*)$ so that consumption jumps by

$$
\begin{aligned}
\Delta c_\tau &= \Lambda W_{\tau+}^* - \left(V'(W_t^*)\right)^{-\frac{1}{R}} \\
&= \Lambda\left(\alpha W_\tau^* - I\right) - \left(V_1'(W_t^*)\right)^{-\frac{1}{R}} = (1 - \alpha^{-\frac{1}{R}})\Lambda\left(\alpha W_\tau^* - I\right)
\end{aligned}
$$

where the second equality follows from the smooth pasting condition. This jump in consumption induces a jump in the state price density given by

$$
\Delta \xi_\tau = e^{-\rho t}\left[u'(c_{\tau+}) - u'(c_\tau)\right] = e^{-\rho t} u'(c_\tau)\left(\frac{1-\alpha}{\alpha}\right) < 0.
$$

Now, assume that the value is three times continuously differentiable (it is possible to check that this is indeed the case when $\rho = 0$) and denote the equilibrium risk premium at time t by θ_t. Using the fact that the pricing kernel also satisfies

$$
d\xi_t = -\xi_t \frac{dB_t}{B_t} - \xi_t \theta_t dZ_t,
$$

we can apply Itô's lemma to get the following result.

Proposition 3 *There exists a competitive equilibrium with dynamically complete markets for the economy with growth options. In this economy, there is a singularity in the pricing kernel and the bond price satisfies*

$$
B_t = [1 + 1_{t>\tau}(e^{1-\frac{1}{\alpha}})]\exp\left\{\int_0^t r_s ds\right\},
$$

where

$$
r_t = \rho + A_t\left[\mu W_t^* - c(W_t^*) - 0.5 P_t \sigma^2 (W_t^*)^2\right],
$$

and where the absolute risk aversion in wealth of the representative consumer A_t and the absolute prudence P_t are respectively defined by

$$A_t = -\frac{V''(W_t^*)}{V'(W_t^*)}, \ and \ P_t = -\frac{V'''(W_t^*)}{V''(W_t^*)}.$$

In addition, the equilibrium risk premium satisfies

$$\theta_t = \sigma W_t^* A_t.$$

Proposition 3 generates several interesting asset pricing implications. These implications are grouped in three categories as follows.

5.1.1. Risk aversion.

Consider first the relative risk aversion in wealth. In the CIR economy, there is no growth option and, under our assumptions, the relative risk aversion in wealth of the representative consumer is constant, given by R. In our model, the relative risk aversion decreases in a significant manner as the wealth level increases to w_i. This decrease in RRA is due to the fact that the growth option goes more and more in the money as the wealth level increases. A consequent action for the consumer to curtail the flow rate of consumption relative to the long-run mean rate of consumption to improve future wealth level. Thus, we obtain time varying risk aversion by virtue of the presence of growth option – this comes from the production side of the economy as opposed to models in which habit formation produces time varying risk aversion to help explain aggregate stock market behavior [see Sundaresan (1989), Constantinides (1990) and Cochrane and Campbell (1999)]. The rate at which the RRA drops is a function of the scale of the growth option. These effects are described in Figure 5 below.

[Insert Figure 5 Here]

5.1.2. Risk premium and interest rates.

As in previous models, the equilibrium risk premium in our model is simply equal to the relative risk aversion in wealth times the volatility of the growth rate of the stock of capital. Thus, the required risk premium is higher if the representative consumer is more risk averse and the model generates a countercyclical risk premium. In contrast

to the CIR economy, equilibrium interest rates in an economy with growth options is not constant but rather depends on the position of the stock of capital with respect to the investment threshold. Again, as soon as the growth option is exercised, our economy reverts to the CIR economy and the equilibrium risk -free rate equals $r^0 = \mu - \sigma^2 R$.

5.1.3. The discounted case.

Similar results obtain in the discounted case where $\rho > 0$. For example, we plot in figure 6 the relative risk aversion the representative consumer as a function his wealth for different values of the option to expand.

[Insert Figure 6 Here]

As in the case without discounting, we find that the relative risk aversion decreases in a significant manner as the wealth level increases to the investment boundary w_i. Again, we obtain time varying risk aversion in our model by virtue of the presence of growth option. Interestingly, when the economy is on the verge of exercising a major growth option, RRA drops much more rapidly.

5.2 Equity prices

Equity in our model is a portfolio of a long position in the stock of capital, plus a long position in a perpetual option to expand the wealth of the economy by a factor of α at a strike price of I. The value of this growth option depends on the optimal consumption policy which plays the role of dividends and on the equilibrium rate which is time-varying prior to the exercise of the growth option. An immediate consequence is that the model will produce interesting interdependency between Tobin's q, consumption rates and interest rates as the growth option moves in and out of the money.

The decomposition of equity value into the value of assets in place and the value of the growth option allows us to write the price of the stock prior to the exercise of the growth option as

$$\mathcal{E}(W_t) = W_t + E\left[e^{-\rho(\tau - t)} \frac{U'(c(W_\tau))}{U'(c(W_t))} \left\{ (\alpha - 1) W_\tau - I \right\} 1_{\{\tau < \infty\}} \middle| \mathcal{F}_t \right].$$

When there is no discounting, the value of equity can be derived in closed-form. In particular, we have the following result.

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Proposition 4 *Assume that the consumer's utility function is undiscounted. Then, the value of equity is given by*

$$\mathcal{E}(W_t) = W_t + [(\alpha - 1) w_i - I] \left(\frac{c(w_i)}{c(W_t^*)} \right)^{-R}, \qquad 1 \neq R > 0,$$

where $c(W_t^*) = 1_{\{W_t^* \in \mathcal{B}_0\}} c^0(W_t^*) + 1_{\{W_t \in \mathcal{C}\}} (V'(W_t^*))^{-\frac{1}{R}}$.

Proposition 4 provides the value of equity in our setup when $\rho = 0$. Interestingly, when the coefficient of relative risk aversion is greater than 1, the value of equity is very similar to that derived in the baseline model. Specifically, we have

$$\mathcal{E}(W_t) = W_t + [(\alpha - 1) w_i - I] \left(\frac{w_i}{W_t} \right)^{-R}, \qquad R > 1,$$

where w_i is given in Proposition 2. That is, equity value is the sum of the current stock of capital plus the value of the growth option. The value of the growth option is equal to the product of the investment surplus and an adjustment factor that captures the concavity of the utility function of the representative consumer. This expression also shows that the book to market ratio exceeds one by an amount that is determined by (i) the immediate exercise value and (ii) a "multiplier." Moreover, the ratio of the value of the growth option to its immediate exercise value is pinned down by the "multiplier" which depends on the optimal consumption policy and risk aversion.

Consider next equity returns. Let Δ denote the delta of the growth option and Γ denote its gamma. Note that at each instant, equity returns depend on the ratio of market to book $q = \frac{\mathcal{E}}{W}$. In fact, a simple application of Itô's Lemma to the function $\mathcal{E}(W) = W + \mathcal{O}(W)$ gives the following dynamics for equity prices:

$$\frac{d\mathcal{E}}{\mathcal{E}} = \mu_{\mathcal{E}}' dt + \sigma_{\mathcal{E}}' dB \quad \text{for } W < W^* \tag{19}$$

where

$$\mu_{\mathcal{E}}' = \frac{1}{q} \left[(1 + \Delta) \mu + \frac{1}{2} \Gamma \sigma^2 W \right] \text{ and } \sigma_{\mathcal{E}}' = \frac{1}{q} (1 + \Delta) \sigma. \tag{20}$$

It is apparent from the valuation formula for equity that the delta and the gamma of the growth option vary with its moneyness. Thus, the conditional moments of stock returns should depend on the wealth levels as well as on the moneyness of the growth options in the economy. In our model, equity prices are expected to increase as the

option goes deeper in the money as does the volatility of equity price changes. That is, an increase in stock returns due to an increase in the stock of capital W is followed by an increase in expected returns. Thus the model generates return momentum. In figure 7 below, we report the conditional moments of equity returns as a function of the wealth of the representative consumer. As shown in the Figure, the moneyness of the growth option is the key factor which determines the extent to which the book to market ratios influence the conditional moments of equity returns.

[Insert Figure 7 Here]

Sharpe ratios can also be computed for various wealth levels. In our model these ratios will be influenced by the feedback effects on interest rates as well. For the base case CIR model the Sharpe ratio is 0.4, given the parameters chosen in the illustrations.

[Insert Figure 8 Here]

Sharpe ratio peaks at about 0.54 when the option to expand three-fold is approaching in-the-money status and then drops rapidly as the option is about to be exercised.

6   Conclusion

This paper develops a general equilibrium model of a production economy which has a risky production technology as well as a growth option to expand the scale of the productive sector of the economy. We show that the presence of the growth option induces the consumer to curtail his consumption to improve the likelihood of investment. This change in the optimal consumption policy is accompanied by an erosion in the option value of waiting to invest. It also has important consequences for the evolution of risk aversion, asset prices and equilibrium interest rates which we characterize in this paper. In particular, we show that the reduction in consumption is accompanied by a fall in the risk aversion in wealth of the representative consumer. Thus, risk aversion displays time variation through the endogenous wealth dependency which arises from the presence of growth options in the economy. This time variation in risk aversion in turn produces a countercyclical, time varying equity risk premium. We also find that the moneyness of the growth option is the key factor which determines the extent to which book to market ratios influence the conditional moments of equity returns.

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Appendix

A. Auxiliary results

In this appendix we recall some results about the central planner's problem absent growth options, establish Lemma 1 and provide a justification for the application of the dynamic programming principle which lead to (15).

In the problem absent growth options, the central planner's wealth process has continuous paths and its dynamics are given by

$$dW_t = W_t\left(\mu dt + \sigma dB_t\right) - c_t dt, \quad W_0 > 0. \tag{21}$$

Let $\mathcal{A}$ denote the corresponding set of admissible consumption plans, that is the set of non negative consumption processes c such that

$$E\left[\int_0^\infty e^{-\rho t}|U(c_t)|dt\right] < \infty. \tag{22}$$

and the solution to equation (21) is non negative throughout the infinite horizon. Given the above definitions, the central planner's problem consists in choosing an admissible consumption plan so as to maximize his lifetime utility. As in the main text, the corresponding value function is denoted by

$$V_0(W_0) := \sup_{c\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right].$$

The following proposition gathers some well known facts about the value function and the optimal strategy absent growth options and will be of repeated use in the proof of our main results.

Proposition 5 *The value function absent growth option is finite if and only if the constant Λ defined by (10) is strictly positive. In this case, we have:*

a) *The value function and the optimal consumption plan absent growth options are respectively given by equations (13) and (12).*

b) *The value function absent growth option is strictly concave, strictly increasing and satisfies $\mathcal{D}V_0 = 0$ on the positive real line.*

c) *The value function absent growth options satisfies the dynamic programming principle in the sense that*

$$V_0(W_{\zeta \wedge \theta}) \geq E\left[e^{-\rho(\zeta-\theta)^+} V_0(W_\zeta) 1_{\{\zeta < \infty\}} + \int_{\zeta \wedge \theta}^{\zeta} e^{-\rho(s-\theta)} U(c_s) ds \,\middle|\, \mathcal{F}_{\zeta \wedge \theta} \right]$$

holds for every admissible consumption plan and all pairs (ζ, θ) of stopping times such that θ is almost surely finite.

Proof. The results of the proposition are well-known and can be found in various references such as Fleming and Soner (1994) or Karatzas and Shreve (1999). ∎

Now consider the problem with growth options. In this case, the central planner's wealth experiences a jump at the time of exercise and evolves according to

$$W_t = W_0 + \int_0^t W_s \left(\mu ds + \sigma dB_s \right) - \int_0^t c_s ds + 1_{\{t > \tau\}} \left[(\alpha - 1) W_\tau - I \right]. \qquad (23)$$

As inn the main text we denote by Θ the corresponding set of admissible consumption and exercise plans, that is the set of pairs (τ, c) such that (22) holds and the solution to equation (23) is non negative throughout the infinite horizon. The central planner's problem consists in choosing an admissible consumption and exercise plan so as to maximize his lifetime utility and the corresponding value function is denoted by

$$V(W_0) = \sup_{(c,\tau) \in \Theta} E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \right].$$

Before justifying the formal application of the dynamic programming principle which lead to equation (15), we start by establishing the validity of Lemma 1.

Proof of Lemma 1. Since the pair (τ, c) with $\tau = \infty$ is an admissible consumption and exercise plan for all $c \in \mathcal{A}$ we have

$$V(W_0) \geq \sup_{c \in \mathcal{A}} E\left[\int_0^\infty e^{-\rho t} U(c_t) dt \right] = V_0(W_0).$$

On other hand, consider the case where the exercise price of the growth option is zero and let $F \geq V$ denote the corresponding value function. In this case, it is optimal to exercise the option immediately and it follows that

$$V(W_0) \leq F(W_0) = V_0(\alpha W_0).$$

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Putting together the two previous inequalities and using the positive homogeneity of the function V_0 we deduce that the finiteness of V is equivalent to that of V_0 and the desired result now follows from the first part of Proposition 5. ∎

The last result in this section establishes some basic properties of the value function and justifies the application of the dynamic programming principle which lead to (15).

Lemma 2 *Assume that the constant Λ is strictly positive, then the central planner's value function is increasing, concave and satisfies* (15).

Proof. The first part of the statement being an easy consequence of (23) and the properties of the utility function we omit the details and turn to the second part.

Let $(c, \tau) \in \Theta$ denote an admissible consumption and exercise plan. Using the continuity of the central planner's wealth after the exercise time in conjunction with the second part of Proposition 5, we obtain

$$E\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right] \leq E\left[e^{-\rho\tau}V_0(W_{\tau+})1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t}U(c_t)dt\right]$$

$$= E\left[e^{-\rho\tau}V_1(W_\tau)1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t}U(c_t)dt\right]$$

where the equality follows from the definition of the reward function and the fact that $W_{\tau+} = \alpha W_\tau - I$. Taking the supremum over the set Θ on both sides of the above expression we conclude that

$$V(W_0) \leq F(W_0) := \sup_{(\tau,c)\in\Theta} E\left[e^{-\rho\tau}V_1(W_\tau)1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t}U(c_t)dt\right].$$

To establish the reverse inequality, let $(\tau, c) \in \Theta$ be arbitrary and consider the non negative consumption process defined by

$$\bar{c}_t := c_t 1_{\{\tau \geq t\}} + c_t^0 1_{\{\tau < t\}}.$$

As is easily seen, the pair $(\bar{c}, \tau)$ belongs to the set Θ of admissible consumption and exercise plans over the infinite horizon and hence satisfies

$$V(W_0) \geq E\left[\int_0^\infty e^{-\rho t}U(\bar{c}_t)dt\right]. \tag{24}$$

On the other hand, using the continuity of the central planner's wealth after the exercise time in conjunction with the optimality of c^0 for the problem absent growth options we obtain

$$E\left[\int_0^\infty e^{-\rho t}U(\bar{c}_t)dt\right] = E\left[e^{-\rho\tau}V_1(W_\tau)1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t}U(c_t)dt\right].$$

Plugging the above expression into equation (24) and taking the supremum over the set Θ on both sides of the resulting expression we conclude that $V \leq F$ holds and our proof is complete. ■

B. Proof of Theorem 1

In order to establish the validity of Theorem 1, we will proceed in several steps. Recall that $\mathcal{D}$ denotes differential operator defined by equation (16) and let

$$V_*(w) := \max\{V_0(w), V_1(w)\}$$

denote the obstacle above which the central planner's value function must lie. In order to get a handle on the central planner's value function, and since there is no way to ascertain its regularity a priori, we start by showing that it solves the obstacle problem (17) in the viscosity sense.

Lemma 3 *Assume that the conditions of Lemma 1 hold. Then the central planner's value function is a viscosity solution of the obstacle problem* (17)*, that is*

a) *Supersolution: For any w_0 and any twice continuously differentiable function φ such that w_0 is a local minimum of the function $V - \varphi$ we have*

$$\min\{\varphi(w_0) - V_*(w_0), \mathcal{D}\varphi(w_0)\} \geq 0.$$

b) *Subsolution: For any w_0 and any twice continuously differentiable function φ such that w_0 is a local maximum of the function $V - \varphi$ we have*

$$\min\{\varphi(w_0) - V_*(w_0), \mathcal{D}\varphi(w_0)\} \leq 0.$$

Proof. The proof of the above result is based on dynamic programming arguments and Itô's lemma. For brevity of exposition, and since a similar result can be found in Morimoto (2003), we omit the details. ■

Relying on the property that the central's planner's value function is a concave viscosity solution of the obstacle problem (15), we can now establish that it is in fact continuously differentiable.

Lemma 4 *Assume that the conditions of Lemma 1 hold. Then the value function of the central planner is continuously differentiable on $(0, \infty)$.*

Proof. Since the central planner's value function is concave and increasing, its left and right derivatives $V'_-(w)$ and $V'_+(w)$ exist and satisfy

$$0 \leq V'_-(w) \leq V_+(w), \qquad w \in (0, \infty).$$

Now, suppose that it is not continuously differentiable so that the above inequality is strict for some w_0. Let $k \in [V'_-(w_0), V'_+(w_0)]$ be arbitrary, fix a strictly positive ε and consider the twice continuously differentiable function

$$\varphi_\varepsilon(w) := V(w_0) + k(w - w_0) - \frac{1}{2\varepsilon}(w - w_0)^2.$$

Using the concavity of the value function, we easily deduce that the point w_0 is a local maximum of the function $V - \varphi_\varepsilon$ and since $V \geq V_*$ by definition, it follows from the viscosity subsolution property of the value function that

$$\mathcal{D}\varphi_\varepsilon(w_0) = \rho V(w_0) - \mu w_0 k + \frac{\sigma^2 w_0^2}{2\varepsilon} - U^*(k) \leq 0.$$

Choosing ε small enough leads to a contradiction and establishes the continuous differentiability of the central planner's value function. ■

As in the text, let $\mathcal{C}$ denote the set of points where the central planner's value function lies strictly above the obstacle, that is

$$\mathcal{C} := \{w \in (0, \infty) : V(w) > V_*(w)\},$$

and let $\mathcal{B} := (0, \infty) \backslash \mathcal{C}$ denote the set of points where it coincides with the obstacle. The next lemma establishes some basic properties of $\mathcal{C}$.

Lemma 5 *Assume that the conditions of Lemma 1 hold true, then there are non negative constants $w_c < I/(\alpha - 1) := w_e < w_i \leq \infty$ such that $\mathcal{C} = (w_c, w_i)$.*

Proof. Using the concavity of the central planner's value function it is easily deduced that $\mathcal{C}$ is a convex and open subset of $(0, \infty)$. It follows that we have $\mathcal{C} = (w_c, w_i)$ for some positive $w_c < w_i < \infty$. In order to show that $w_c \leq w_e \leq w_i$ and thus complete the proof, assume that it is not the case. Then it follows from the definition of $\mathcal{C}$ that we have $w_e \in \text{int}(\mathcal{B})$ which, together with the definition of the obstacle, contradicts the continuous differentiability of the central planner's value function. ∎

Using the definition of the set $\mathcal{B}$ in conjunction with the above result, we have that the state space can be decomposed as $\mathcal{B}_0 \cup \mathcal{C} \cup \mathcal{B}_1$ where

$$\mathcal{B}_0 := \{w > 0 : V(w) = V_0(w)\} = (0, w_c],$$
$$\mathcal{B}_1 := \{w > 0 : V(w) = V_1(w)\} = [w_i, \infty).$$

In particular, since $w_e \notin \text{int}(\mathcal{B})$ we have that the central planner's value function is twice continuously differentiable on $(0, w_c) \cup (w_i, \infty)$ and all there remains to establish in order to prove that the central planner's value function is a classical solution of the obstacle problem is that it is twice continuously differentiable on the set $\mathcal{C}$. This is the content of the next lemma.

Lemma 6 *Assume that the conditions of Lemma 1 hold, then the central planner's value function is twice continuously differentiable on the set $\mathcal{C} \cup \mathcal{B}_0 = (0, w_i)$*

Proof. Using the definition of $\mathcal{C}$ in conjunction with the fact that the value function is viscosity solution of the obstacle problem, it is easily deduced that V is also a viscosity solution of the equation

$$\mathcal{D}\varphi(w) = 0, \qquad w \in \mathcal{C} = (w_c, w_i). \tag{25}$$

Since the central planner's value function is once continuously differentiable on the strictly positive real line, the function defined by

$$h(w) := \rho V(w) - \mu w V'(w) - U^*(V'(w))$$

is continuous on $\mathcal{C}$. Now, fix an arbitrary real number ε and consider the ordinary second order differential equation

$$-\tfrac{1}{2}\sigma^2 w^2 \varphi_\varepsilon''(w) + h(w) - \varepsilon = 0, \tag{26}$$

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Let $(\check{w}, \hat{w})$ with $\check{w} \leq w_e \leq \hat{w}$ be a non degenerate open interval contained in $\mathcal{C}$ and, for any point in this interval, define the functions

$$\theta_\varepsilon(w) := 2 \int_{\check{w}}^w d\xi \int_{\check{w}}^\xi dr \left(\frac{h(r) - \varepsilon}{\sigma^2 r^2} \right),$$

$$\varphi_\varepsilon(w) := V(\check{w}) + \theta_\varepsilon(w) + (w - \check{w}) \left(\frac{V(\hat{w}) - V(\check{w}) + \theta_\varepsilon(\hat{w})}{\hat{w} - \check{w}} \right).$$

Then φ_ε is a classical solution of equation (26) on the set $\mathcal{O} := (\check{w}, \hat{w})$ and is equal to the central planner's value function on the boundary of this set. In order to compare the functions V and φ_ε we use the fact that the central planner's value function is a viscosity solution of equation (25). Assume that ε is strictly positive and that the function $V - \varphi_\varepsilon$ has a a local maximum at some point $w_0 \in \mathcal{O}$. Then the viscosity subsolution property implies that

$$\mathcal{D}\varphi_\varepsilon(w_0) = h(w_0) - \tfrac{1}{2}\sigma^2 w^2 \varphi_\varepsilon''(w_0) \leq 0$$

which contradicts the fact that φ_ε is a solution to equation (26) on the set $\mathcal{O}$. Thus, the function $V - \varphi_\varepsilon$ attains its maximum over $\mathrm{cl}(\mathcal{O})$ at the boundary of this set and, since it is equal to zero there, it follows that we have

$$V(w) \leq \varphi_\varepsilon(w), \qquad w \in \mathrm{cl}(\mathcal{O}).$$

Letting ε decrease to zero on both sides we conclude that $V \leq \varphi_0$ on $\mathcal{O}$. Similarly, letting ε be negative and using the fact that the central planner's value function is a viscosity supersolution of (25) we obtain that $\varphi_0 \leq V$ holds and conclude that the function $V = \varphi_0$ is twice continuously differentiable on the set $\mathcal{O}$. From the arbitrariness of the constants $\check{w}$ and $\hat{w}$ it is now clear that the central planner's value function is twice continuously differentiable on the set $\mathcal{C}$.

In order to complete the proof it is sufficient to show that the central planner's value function is twice continuously differentiable at the point w_c. This easily follows from the continuous differentiability of V and the fact that the value function absent growth options satisfies $\mathcal{D}V_0 = 0$ on the positive real line, we omit the details. ∎

Putting together the results of Lemma 1–3, 5 and 6 we have proved that the value function is a classical, concave solution of the obstacle problem (17) such that

$$\varphi(w) \leq V_0(\alpha w), \qquad w \in (0, \infty). \tag{27}$$

In order to show that it is in fact the unique such solution of the obstacle problem, we will rely on the following two lemmas.

Lemma 7 *Assume that the conditions of Lemma 1 hold true, let φ be a classical, concave solution of the obstacle problem such that equation (27) is satisfied. Set*

$$c_\varphi(w) := \left(\varphi'(w)\right)^{-\frac{1}{R}}$$

and denote by $w_\varphi > w_e$ the investment threshold associated with the function φ. Then, for any initial condition in the interval $(0, w_\varphi)$, the stochastic differential equation

$$W_t = W_0 + \int_0^t 1_{\{W_s < w_\varphi\}} \left[(\mu W_s - c_\varphi(W_s))\, ds + \sigma W_s dZ_s\right] \tag{28}$$

admits a unique, non exploding solution which is positive and strictly positive if the central planner's relative risk aversion is larger than one.

Proof. Since the function φ is a classical solution of the obstacle problem, it is twice continuously differentiable on the set

$$\mathcal{H} := (0, w_\varphi) = \{w : \varphi(w) > V_1(w)\} \cup \{w : \varphi(w) = V_0(w)\}$$

and it follows that c_φ is locally Lipschitz continuous when restricted to the set $\mathcal{H}$. On the other hand, using (27) in conjunction with the fact that $\varphi \geq V_0$, l'Hopital's rule and the expression of the value function absent growth options we deduce that

$$c_\varphi(0) = \lim_{w \to 0} \varphi'(w)^{-\frac{1}{R}} = 0.$$

Since the function c_φ is locally Lipschitz continuous when restricted to $\mathcal{H}$ and the coefficients the stochastic differential equation are equal to zero on the boundary of this set, it follows from standard results that (28) admits a unique strong solution which is absorbed upon reaching the boundary of the set $\mathcal{H}$.

If the investment threshold is finite, then the solution is obviously non exploding, so assume that $w_\varphi = \infty$. Using the non negativity of c_φ in conjunction with standard comparison theorems for solutions of stochastic differential equations we deduce that

$$W_t \leq W_t^0 := W_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma B_t\right], \quad t \in (0, \infty). \tag{29}$$

The process on the right hand side being a geometric Brownian motion, it is almost everywhere finite and it follows that the solution to (28) is non exploding.

Let T_0 denote the first time that the solution hits the origin and define another stopping time by setting $S := \tau_\varphi \wedge T_0$. In order to complete the proof we need to show that, in the case where the central planner's coefficient of relative risk aversion is strictly larger than one, the event

$$\Delta := \{T_0 < \infty\} = \{\omega \in \Omega : \exists t \in (0, \infty) \text{ s.t. } W_t = 0\}$$

has zero measure. Using the fact that the function φ is a classical solution to the obstacle problem in conjunction with the second assertion of Proposition 5 we obtain that the equality $\mathcal{D}\varphi = 0$ holds on $\mathcal{H}$ and it now follows from Itô's lemma that

$$Y_t = e^{-\rho t \wedge S}\varphi(W_{t \wedge S}) + \int_0^{t \wedge S} e^{-\rho s}U(c_\varphi(W_s))ds \tag{30}$$

is a local martingale. If the relative risk aversion of the central planner is strictly larger than one, then the functions (V_0, U) are both negative and since φ satisfies equation (27) we conclude that the local martingale Y is negative, and hence a global submartingale by the reverse Fatou lemma. In particular,

$$\varphi(W_0) \leq E\left[e^{-\rho S}\varphi(W_S)1_{\{\Delta\}}\right] \leq E\left[e^{-\rho T_0}V_0(\alpha W_{T_0})1_{\{\Delta\}}\right].$$

where the second inequality follows from equation (27) and the fact that $S = T_0$ holds almost surely on A. Since the left hand side of the above expression is finite and the value function absent growth options equals $-\infty$ at zero we conclude that Δ is a null set and our proof is complete. $\blacksquare$

Lemma 8 *Assume that the conditions of Lemma 1 hold true and let the function φ be as in the statement of Lemma 7. Then we have*

$$\varphi(W_0) = E\left[e^{-\rho \tau_\varphi}V_1(W_{\tau_\varphi})1_{\{\tau_\varphi < \infty\}} + \int_0^{\tau_\varphi} e^{-\rho s}U(c_\varphi(W_s))ds\right]$$

where the stopping time τ_φ is the first time that the solution to equation (28) reaches the strictly positive investment threshold $w_\varphi := \inf\{w : \varphi(w) = V_1(w)\}$.

Proof. In order to establish the desired result, we start by showing that the local martingale defined by (30) is a uniformly integrable martingale. Using equation (27)

in conjunction with the fact that $\varphi \geq V_0$ it is easily deduced that Y is either always positive or always negative. Fatou's lemma thus implies that

$$|\varphi(W_0)| = |Y_0| \geq E|Y_S| = E\left[e^{-\rho S}|\varphi(W_S)| + \int_0^S e^{-\rho t}U(c_\varphi(W_t))dt\right]$$
$$\geq E\left[\int_0^S e^{-\rho t}|U(c_\varphi(W_t))|dt\right]$$

and, since the left hand side of the above expression is finite, Lebesgues's dominated convergence theorem allows us to conclude that the finite variation process

$$B_t := \int_0^{t \wedge S} e^{-\rho t}U(c_\varphi(W_t))dt$$

is uniformly integrable. Now consider the process defined by $A := Y - B$. Using the assumptions of the lemma in conjunction with (29), it is easily deduced that

$$|A_t| \leq D_t := e^{-\rho t \wedge S}\left|V_0\left(\beta W_{t \wedge S}^0\right)\right| \tag{31}$$
$$= D_0 \exp\left[\sigma(1-R)B_{t \wedge S} - \int_0^{t \wedge S}\left(\Lambda R + \tfrac{1}{2}\sigma^2(1-R)^2\right)du\right]$$

holds for some strictly positive constant β where the last equality follows from the first assertion of Proposition 5, Itô's lemma and equation (10). The marginal propensity to consume being strictly positive, D is uniformly integrable and it follows that A is uniformly integrable. The local martingale Y being the sum of two uniformly integrable process it is a uniformly integrable martingale. In particular,

$$\varphi(W_0) = \lim_{t \to \infty} E[Y_t] = E\left[\lim_{t \to \infty} Y_t\right] = E\left[\lim_{t \to \infty} A_t + B_S\right]$$

and all that remains to establish is that

$$\lim_{t \to \infty} A_t + B_S = e^{-\rho \tau_\varphi}\varphi(W_{\tau_\varphi})1_{\{\tau_\varphi < \infty\}} + \int_0^{\tau_\varphi} e^{-\rho s}U(c_\varphi(W_s))ds. \tag{32}$$

Consider the integral term first and assume that the central planner's relative risk aversion is smaller than one for otherwise the desired result follows from Lemma 7. By definition of the stopping time S we have

$$B_S = 1_{\{T_0 < \infty\}}B_{T_0} + 1_{\{T_0 = \infty\}}B_{\tau_\varphi} = B_{\tau_\varphi} + 1_{\{T_0 < \infty\}}\left[B_{T_0} - B_{\tau_\varphi}\right],$$

and using the fact that the solution to the stochastic differential equation (28) is equal to zero on $[\![T_0, \infty[\![$ in conjunction with the fact that $U(0) = 0$ when the central planner's relative risk aversion is smaller than one, we conclude that $B_S = B_{\tau_\varphi}$.

Now consider the process A. Using the result of Lemma 7 in conjunction with the assumptions of the statement it is easily deduced that

$$\lim_{t \to \infty} A_t = 1_{\{S < \infty\}} A_S + \lim_{t \to \infty} 1_{\{S = \infty\}} A_t = 1_{\{\tau_\varphi < \infty\}} A_{\tau_\varphi} + \lim_{t \to \infty} 1_{\{S = \infty\}} A_t.$$

Using equation (31) in conjunction with the fact that a geometric Brownian motion with negative drift converges to zero, we obtain that the last term on the right hand side of the above expression is equal to zero and it follows that (32) holds. ∎

As explained after the statement of Theorem 1, the optimal investment strategy consists in exercising the growth option as soon as the central planner's wealth enters the region $\mathcal{B}_1 = [w_i, \infty)$. The following result shows that the investment threshold w_i is finite and hence that there always exists an optimal exercise time.

Corollary 1 *Assume that all the conditions of Lemma 7 hold true, then w_φ is finite. In particular, the central planner's investment threshold is finite.*

Proof. Assume towards a contradiction that the investment threshold w_φ is infinite. Since the solution to the stochastic differential equation (28) is non explosive this implies that $\tau_\varphi = \infty$ and it now follows from Lemma 8 that

$$\varphi(W_0) = E\left[\int_0^\infty e^{-\rho t} U(c_\varphi(W_t)) dt\right]$$

The function φ being either always positive or always negative depending on the sign of the utility function, this implies that $c_\varphi \in \mathcal{A}$ and it follows that $\varphi \leq V_0$. Since the set of points where $V_0 < V_1$ is non empty, this contradicts the assumptions of the statement and hence completes our proof. ∎

In order to complete the proof of Theorem 1, all there remains to establish is the optimality of the pair (c^*, τ^*) and the fact that the central planner's value function is the unique classical, concave solution of the obstacle problem such that (27) holds. This is the content of the following verification lemma.

Lemma 9 *Assume that φ is a classical, concave solution to the obstacle problem such that equation (27) holds true and define the consumption and exercise plan*

$$c_t^* := 1_{\{t \leq \tau^*\}} c_\varphi(W_t^*) + 1_{\{t > \tau^*\}} c^0(W_t^*), \qquad \tau^* := \tau_\varphi = \inf\{t : W_t^* = w_\varphi\}.$$

where the process W^* denotes the corresponding solution to the stochastic differential equation (23). Then the function φ coincides with the central planner's value function and the admissible consumption and exercise plan (c^*, τ^*) is optimal.

Proof of Lemma 9. Let the function φ satisfy the assumptions of the statement, fix an arbitrary pair $(\tau, c) \in \Theta$ and consider the process

$$G_t := e^{-\rho(t \wedge \tau)} \varphi(W_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\rho s} U(c_s) ds$$

where the process W is the solution to (23) associated with the pair (c, τ). Using Itô's lemma in conjunction with the that the function φ is a classical solution of the obstacle problem we obtain that the process G is a local supermartingale. On the other hand, using the fact that $\varphi \geq V_0$ in conjunction with the third assertion of Proposition 5 and the admissibility of the pair (c, τ) we obtain that

$$
\begin{aligned}
G_t &\geq e^{-\rho(t \wedge \tau)} V_0(W_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\rho s} U(c_s) ds \\
&\geq E\left[e^{-\rho \tau} V_0(W_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-\rho s} U(c_s) ds \,\middle|\, \mathcal{F}_{t \wedge \tau} \right] \\
&\geq \Gamma - E\left[\int_0^\tau e^{-\rho s} |U(c_s)| ds \,\middle|\, \mathcal{F}_t \right]
\end{aligned}
$$

holds for some constant Γ. Since the consumption plan c satisfies (22) the process on the right hand side of the above expression is a uniformly integrable martingale and it follows from Fatou's lemma that G is a supermartingale. In particular, we have

$$\varphi(W_0) \geq E\left[\lim_{t \to \infty} G_t \right] = E\left[\lim_{t \to \infty} e^{-\rho(t \wedge \tau)} \varphi(W_{t \wedge \tau}) + \int_0^\tau e^{-\rho s} U(c_s) ds \right]. \qquad (33)$$

Using the assumptions of the statement in conjunction with an argument similar to that which lead to equation (31) we deduce that

$$\lim_{t \to \infty} 1_{\{\tau = \infty\}} e^{-\rho t} |\varphi(W_t)| \leq \lim_{t \to \infty} 1_{\{\tau = \infty\}} e^{-\rho t} \left| V_0\left(\beta W_t^0 \right) \right| = 0$$

where the last equality follows from the fact that a geometric Brownian motion with negative drift converges to zero. Plugging this back into equation (33) and using the fact that φ lies above the obstacle, we obtain that

$$\varphi(W_0) \geq E\left[e^{-\rho\tau}V_1(W_\tau)1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho s}U(c_s)ds\right]$$

Taking the supremum with respect to the set of admissible consumption and exercise plans on both sides of the above expression we conclude that $\varphi \geq V$.

In order to establish the reverse inequality, assume that the central planner's initial wealth lies in $(0, w_\varphi)$ for otherwise the result is immediate. Since W^* coincides with the unique solution to the stochastic differential equation (28) up to the exercise time, it follows from Lemma 8 that we have

$$\varphi(W_0) = E\left[e^{-\rho\tau^*}V_1(W_\tau^*)1_{\{\tau^*<\infty\}} + \int_0^{\tau^*} e^{-\rho t}U(c_t^*)dt\right] = E\left[\int_0^\infty e^{-\rho t}U(c_t^*)dt\right]$$

where the second equality is a consequence of equation (23), the definition of the function V_1 and the optimality of c^0 for the problem absent growth options. The left hand side of the above expression being finite, we have that the pair (c^*, τ^*) belongs to the set Θ of admissible plans and it follows that $\varphi \leq V$. In conjunction with the first part, this implies that φ coincides with central planner's value function and establishes the optimality of the pair (c^*, τ^*). ∎

B. Proof of Propositions 2 and 1

In order to establish the validity of Theorem 2, all there is to prove is that the value function given in the statement is a classical, concave solution of the obstacle problem such that equation (27) holds. For brevity of exposition, and since this follows from straightforward algebraic computations, we omit the details.

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Table 1: Investment policy in general equilibrium. This table reports the selected investment threshold W^* assuming that the ratio $(\alpha - 1)/I$ is fixed at 1. The long-run marginal propensity to consume (MPC) out of wealth is shown in parentheses. This MPC is computed using the CIR equilibrium that prevails in our economy after the growth option is exercised.

TABLE 1		$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
	$\rho = 10\%$	1.282 (11.375%)	1.309 (10.50%)	1.346 (9.375%)
$I = 2$	$\rho = 12\%$	1.244 (12.375%)	1.275 (11.50%)	1.303 (10.375%)
	$\rho = 14\%$	1.213 (13.375%)	1.248 (12.50%)	1.278 (11.375%)
	$\rho = 10\%$	1.320 (11.375%)	1.363 (10.50%)	1.414 (9.375%)
$I = 1.1$	$\rho = 12\%$	1.269 (12.375%)	1.316 (11.50%)	1.357 (10.375%)
	$\rho = 14\%$	1.199 (14.375%)	1.279 (12.50%)	1.3224 (11.375%)

Table 2: Investment policy in partial equilibrium. This table reports the selected investment threshold W^* in the baseline model of real options that ignore the feedback effects of investment on equilibrium quantities [such as consumption] and wealth-dependent risk aversion. Again, we assume that the ratio $(\alpha - 1)/I$ is fixed at 1. The long-run marginal propensity to consume out of wealth is shown in parentheses.

TABLE 2	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$\rho = 10\%$	1.934 (11.38%)	2.432 (10.50%)	3.309 (9.38%)
$\rho = 12\%$	1.600 (12.38%)	1.907 (11.50%)	2.362 (10.38%)
$\rho = 14\%$	1.432 (13.38%)	1.248 (12.50%)	1.963 (11.38%)
$\rho = 16\%$	1.333 (14.38%)	1.511 (13.50%)	1.744 (12.38%)

Figure 1: State space of the optimal consumption and investment plans.
The growth option has positive NPV for all $w \geq w_e$. In the continuation region $[w_i, \infty)$, it is optimal to exercise the growth option immediately. In the continuation region $(0, w_i)$, it is optimal to wait.
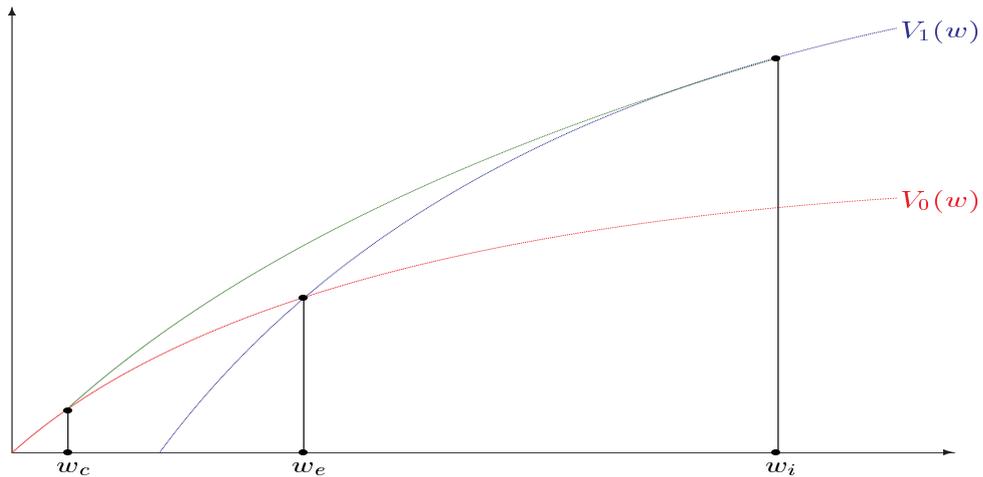

Figure 2: Selected investment threshold. Figure 2 plots the selected investment threshold as a function of the relative risk aversion coefficient R. In this figure we have $\rho = 0$, $\mu = 0.01$, and $\sigma = 0.5$. The solid line represents an environment in which $I = 1.4$. The dashed line represents an environment in which $I = 2$.

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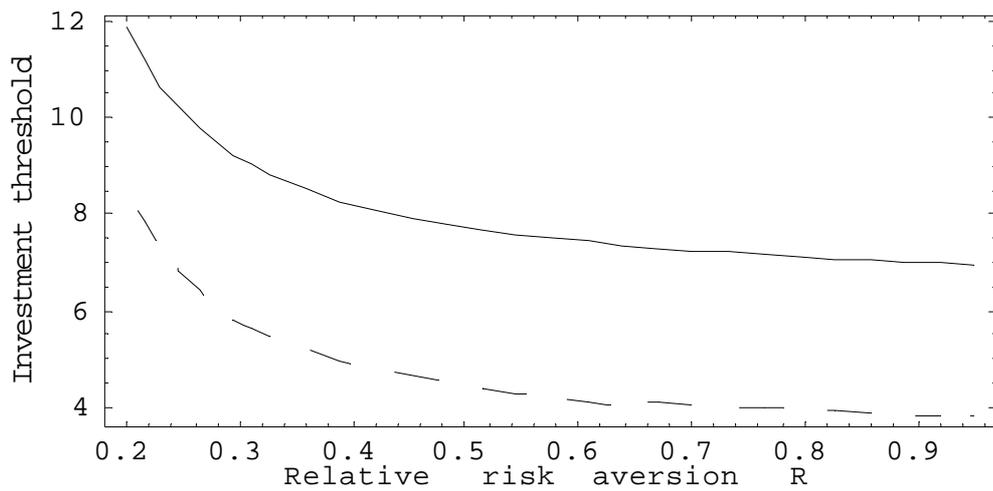
Figure 3: Consumption rate when $\rho = 0$. Figure 3 plots the consumption rate as a function of wealthin the CIR economy (top line) and in the economy with growth options (bottom curve). In this figure we have$\rho = 0$, $\mu = 0.01$, $\sigma = 0.5$, $R = 0.5$ and $w_i = 2.23$.
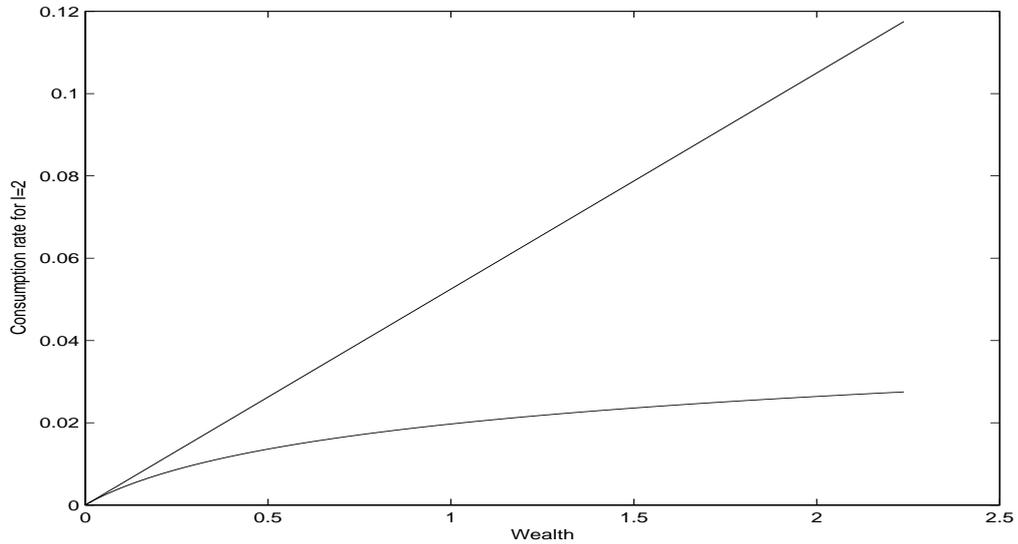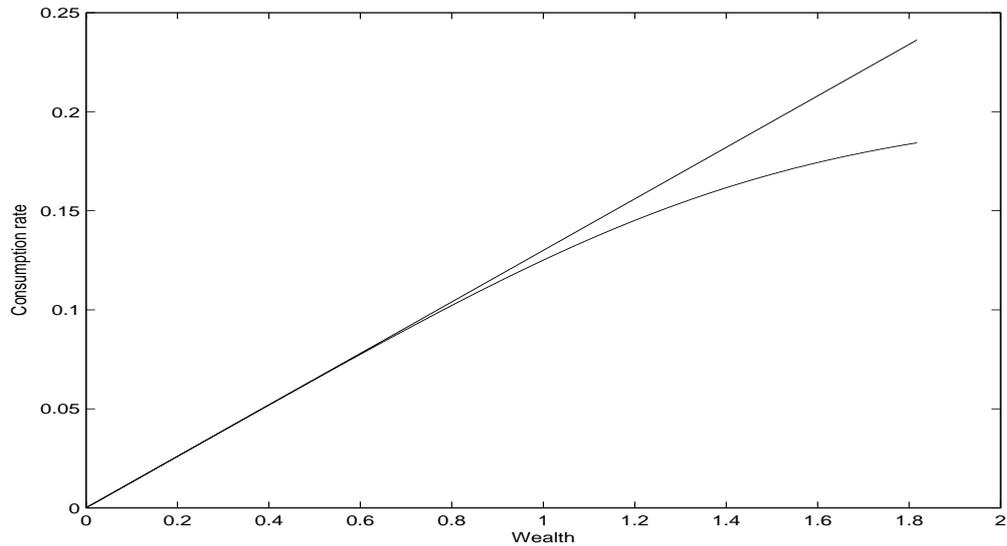

Figure 4: Consumption rate when $\rho > 0$. Figure 4 plots the consumption rate as a function of wealthin the CIR economy (top line) and in the economy with growth options (bottom curve). In this figure we have $\rho = 0.10$, $\mu = 0.10$, $\sigma = 0.2$, and $R = 2$ and $w_i = 1.81$.

Figure 5: Relative risk aversion in wealth when $\rho = 0$. Figure 5 plots the relative risk aversion in wealth as a function of wealth. In this figure we have set the input parameters as follows: $\rho = 0$, $\mu = 0.01$, $\sigma = 0.5$, and $R = 0.5$ and we have $w_i = 2.23$.
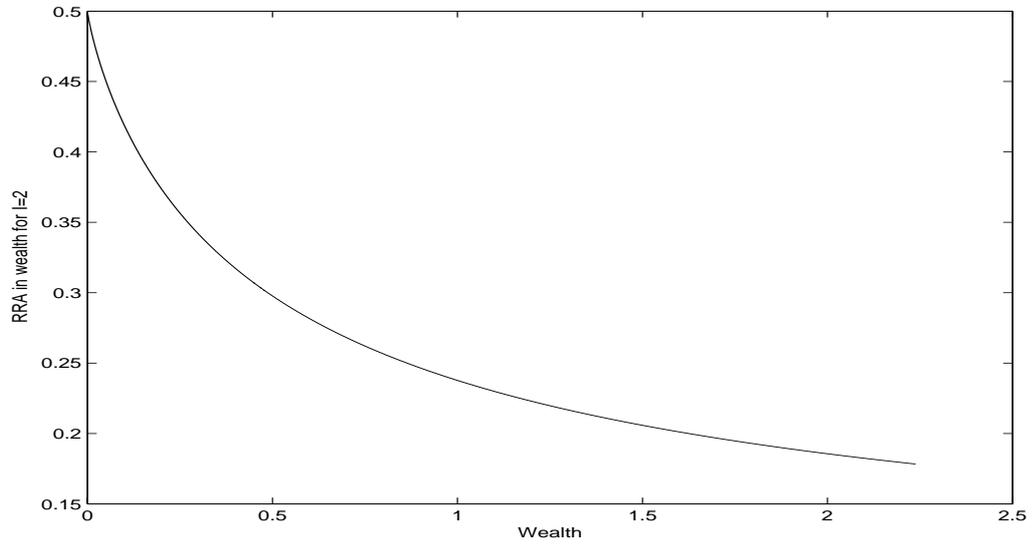

Figure 6: Scaled relative risk aversion in wealth when $\rho > 0$. Figure 6 plots the relative risk aversion in wealth as a function of wealth. In this figure we have set the input parameters as follows: $\rho = 0.10$, $\mu = 0.10$, $\sigma = 0.2$, and $R = 2$ and we have $w_i = 1.81$.
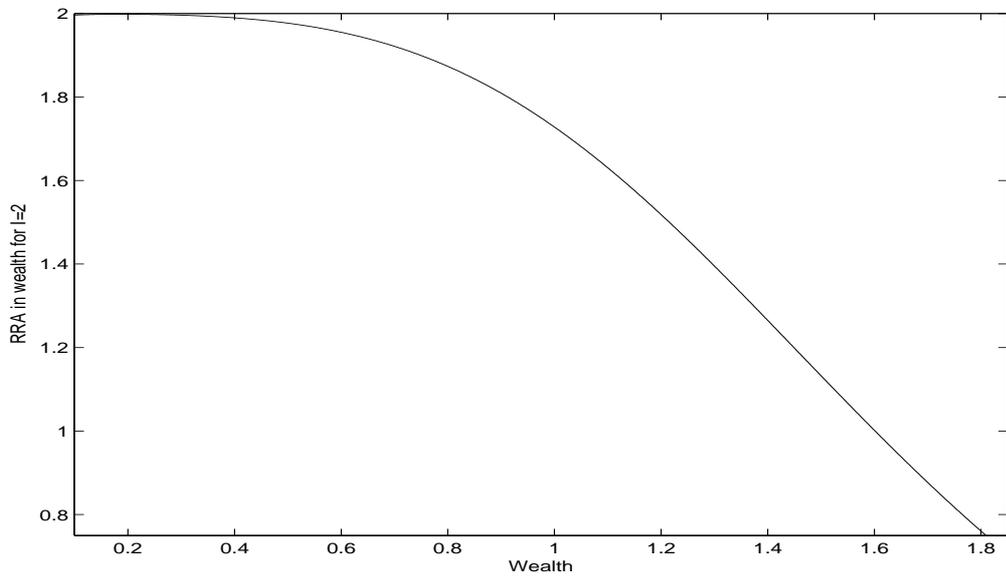
Figure 7: Conditional moments of equity returns when $\rho > 0$. Figure 7 plots the conditional moments of equity returns in the model with growth options. In this figure we have set the input parameters as follows: $\rho = 0.1$, $\mu = 0.1$, and $\sigma = 0.2$ and we have $w_i = 1.81$.
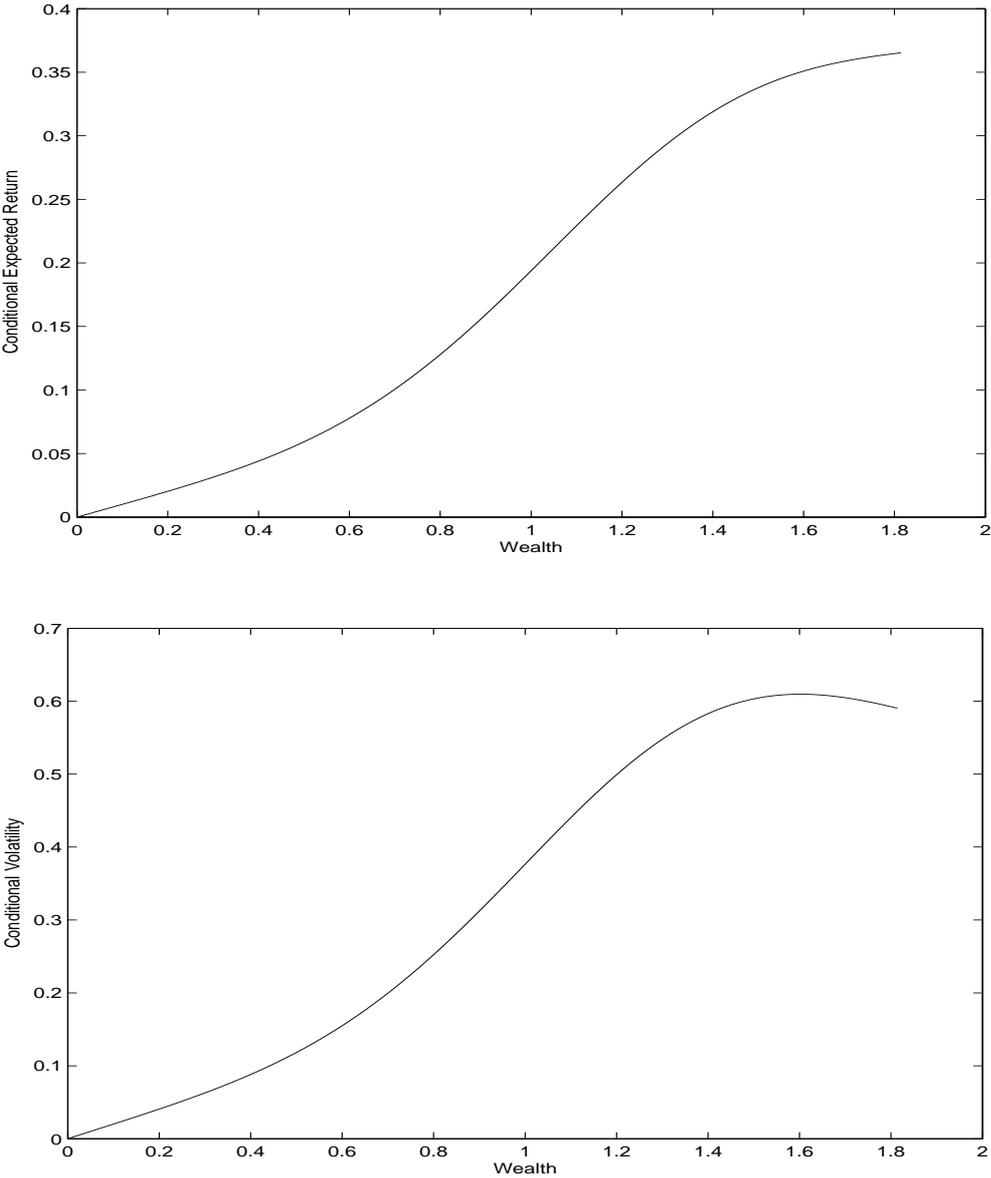
Figure 8: Sharpe ratio when $\rho > 0$. Figure 8 plots the Sharpe ratio in the model with growth options. In this figure we have set the input parameters as follows: $\rho = 0.1$, $\mu = 0.1$, and $\sigma = 0.2$. For the base case CIR model the Sharpe ratio is 0.4.