

A Life Cycle Model of Firm Value

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ABSTRACT: Business valuation must rely on assumptions about the time dynamics of firms' future value flows. By separating the standard net cash flow variable into expenditures and revenues and replacing the commonly assumed constant future growth rate with a life cycle trajectory, one obtains a flexible, non-monotonic description of a firm's evolution through time at the level of its individual investments and their aggregation into intrinsic enterprise value. A parsimonious set of assumptions permits an empirical estimation that requires no inputs other than financial accounting data and that delivers estimates of intrinsic firm value and the cost of capital that align well with observed market values. Empirical tests support the model's analytical predictions about the time-series behavior of various growth rates and profitability ratios.

Keywords: *valuation, investment, life cycle, cost of capital*

JEL Classification: D21, G12, G17, G31, G32

1. Introduction

The valuation of firms and the analysis of their profitability form a central topic in finance, accounting and economics. Changing the unit of analysis from dividends to cash flow and earnings, valuation theory has over time shifted its focus from the receipt and distribution of profits toward the original profit-generating activities.¹ Explicit models of the firm's underlying investment problem have further separated the net profit into investment expenditures and revenues and thereby continued the move toward the source of value creation.² But while strengthening the theoretical connection between value flows and enterprise value, the move toward metrics closer to the origin of value creation does not by itself yield an explanation of the evolution of these value flows through time.³ How to project earnings, cash flows, and investments into the future remains a persistent problem, and both researchers and practitioners tend to rely on a variety of heuristic, firm-specific and often ad-hoc expedients, such as the assumption of a terminal growth rate.⁴ The aim of this paper is to develop a more rigorous theoretical basis for the time dynamics of firms' investing behavior and profitability, based on the well-subscribed idea that businesses transition through characteristic and predictable phases as they mature over the course of their lifespan. This life cycle model yields a parsimonious description of how valuation multiples and various other common financial ratios vary with firm age, and it permits an estimation of intrinsic value and the cost of capital based on the time-series of a few basic accounting variables.

The model combines aspects of neoclassical investment theory, namely, the generation of revenues from successive, accumulative rounds of resource expenditure, with aspects of life cy-

¹ Dividend-based valuation has long been recognized as impractical, as payout rates and capital structure are discretionary and hence distributions over a finite horizon have no definite relation to firm value (Gordon and Shapiro 1956; Modigliani and Miller 1958). Moving to free cash flows as the valuation attribute, one can abstract from forecasting dividends, share repurchases and capital structure choices (Penman and Sougiannis 1998). But while closer to the underlying value-generating process, cash flows over a finite horizon are generally not fully indicative of the value created over that period. See, for example, Penman (2011) for a discussion. More recently, the residual income valuation literature has moved the focus to earnings and book value, which aim explicitly to capture value added over a period and resources invested to generate earnings (Edwards and Bell 1964; Ohlson 1995; Feltham and Ohlson 1995, 1996; Nissim and Penman 2001; Easton 2004; Ohlson and Jüttner-Nauroth 2005; Ohlson and Gao 2006).

² Examples include the capacity models by Rajan and Reichelstein (2009), Abel and Eberly (2011), and Nezlabin, Rajan and Reichelstein (2012, 2014), as well as dynamic investment models (Strebulaev and Whited 2011).

³ See, for example, Wahlen, Baginski and Bradshaw (2008). And, given consistent inputs, all valuation approaches yield, of course, identical results (Lundholm and O'Keefe 2001).

⁴ Models with specific, finite-horizon forecasts and a fixed terminal growth rate include, for example, Frankel and Lee (1998), Easton (2004), Ohlson and Jüttner-Nauroth (2005), and McNichols, Rajan and Reichelstein (2014). An alternative description of firms' time-series behavior appears in the Feltham-Ohlson model (Ohlson 1995; Feltham and Ohlson 1995, 1996), which assumes abnormal earnings to evolve according to a symmetric stochastic process. Later extensions, variants and empirical implementations of this model include, for example, Dechow, Hutton and Sloan (1999); Feltham and Ohlson (1999); Ang and Liu (2001); and Ohlson (2009).

cle theory, namely, the shifting of the expenditure-revenue relationship as a function of firm age. The basic building block of the model is an investment-level revenue function that maps out the payoff trajectory of an investment made at a specific time in the firm's life. The aggregation of these revenues and investment expenditures across time describes the evolution of the firm as a whole. Two critical assumptions determine the properties of this evolution. First, each investment generates revenues according to the well-known product life cycle pattern of introduction-growth-maturity-decline. Second, the life cycles of individual investments begin at more advanced stages as the firm matures, as more experience allows a firm to realize the payoffs produced by new investments increasingly quickly.

The life cycle model describes the firm-level time-series behavior of several financial ratios central to valuation theory and financial statement analysis. The model implies that the growth rates of revenues and investment decrease and may eventually become negative as the firm ages, consistent with the notion that startup firms grow aggressively while mature firms stabilize or decline. The ratio of the value of new investments to the firm's contemporaneous revenue decreases in firm age, i.e., young firms collect only small amounts of revenue relative to the value of the assets they put in place, whereas the converse applies to mature firms, who realize more value than they simultaneously deploy. A related implication is that a firm's market-to-sales ratio, i.e., the ratio of intrinsic firm value to its contemporaneous revenues, decreases monotonically over time. Empirical tests in section 4 show results consistent with these predictions.

To connect the analysis to accrual-basis financial metrics, the model features a matching-principle-based accounting rule in which investment expenditures are capitalized and subsequently expensed in proportion to the revenues they generate over time. If the model's premise is correct that investment life cycles condense as the firm ages, this accounting rule implies a decreasing ratio of the net-to-gross carrying amount of depreciable assets on firms' balance sheets over time, and the empirical results in section 4 show supporting evidence. Further, the accounting rule implies a return on assets that is monotonically increasing in firm age and thus comonotonic with the profitability of the firm's contemporaneous new investments. Notably, however, neither the return on sales nor the market-to-book ratio generally have this monotonicity property, and the empirical test results indeed suggest that the period-to-period changes in return on sales (market-to-book) are lower than the changes in return on assets (market-to-sales). Overall, the

analytical properties of the model underline the importance of firm age and maturity in comparing and interpreting financial metrics, both across time and across entities.

Empirical implementation of the model requires only firms' own financial statement data. The availability of analyst reports or of industry or peer firm data is not a limiting factor because earnings forecasts and growth prospects follow directly from the model dynamics. The model's lead-lag relationship between expenditures and revenues can rationalize temporarily negative net cash flow while also mapping out the path to positive future cash flow. Ad-hoc assumptions about long-run profitability or the outright elimination of cash-flow-negative firms from test samples is not required. Section 4 of the paper illustrates one possible estimation approach, based on revenues, assets and earnings. The results show a log-correlation between model-implied firm values and actual contemporaneous market values of 0.92 across a large sample of 10,805 firms.

The life cycle model assumes no particular asset pricing theory but, like neoclassical models such as Abel and Eberly (2011) and Nezlobin, Rajan and Reichelstein (2012, 2014), posits the existence of a long-term, stable cost of capital by which investors discount the firm's cash flows and which managers consider in their investing decisions. Since the model's investment-level value function not only maps out future revenues but also measures their present value, one can extract this discount rate jointly with intrinsic value from the time series of revenues, expenditures, and related financial statement variables. The empirical test in section 4 produces a weighted-average cost of capital with a mean of just under 10% across all firms in a sample period between 1950 and 2017, consistent with average historical returns. Previous research has generally faced the tradeoff between estimating firm value from a given cost of capital, such as Frankel and Lee (1998), and rationalizing the cost of capital based on observed stock market values, such as the implied cost of capital estimation by Gebhardt, Lee and Swaminathan (2001).⁵

Two strands of literature each partially inform the model: neoclassical investment theory and the life cycle of the firm. The life cycle model is compatible with neoclassical investment-based valuation methods in its conception of the firm as a series of overlapping investment decisions, albeit with a different emphasis. Neoclassical models center on characterizing a firm's investment problem in terms of its production technology and demand curve, and on identifying the corresponding optimal operating strategy in terms of deploying and adjusting productive capital

⁵ An exception is the expected-rate-of-return estimation by Christodoulou, Clubb and Mcleay (2016), which extracts a rate estimate from accounting data based on the linear information dynamics of the Feltham-Ohlson model.

(e.g., Rajan and Reichelstein 2009; Abel and Eberly 2011; Strebulaev and Whited 2011; Nezlobin, Rajan and Reichelstein 2012, 2014).⁶ How the investment problem changes through time is generally taken as exogenous. The life cycle model instead makes no assumptions about the nature of the firm's investment problem but develops the time-series evolution of actual investment decisions and outcomes from the idea that maturing firms gain efficiency and experience over time. The rationale for this shift in focus is that the critical property of an empirically implementable valuation model is not the within-period mapping from expenditures to payoffs but rather the time-series trajectory of the realized value flows, which ultimately determine enterprise value. The within-period investment problem can generally not be tested empirically because only the investment choice actually made, along with its subsequent payoff, is observable, while the mapping from all alternative choices to their hypothetical outcomes remains hidden.

The life cycle concept has been operationalized in various ways. Past research has applied the idea both at the level of firms and at the level of products, wherein one can further vary the fineness of the analysis from broad product classes to particular product forms to individual brands (Polli and Cook 1969).⁷ A common view considers a firm's evolution over time as shaped by competitive forces in its industry and, in this vein, has proposed that firms focus early on building excess capacity to deter potential entrants to the industry (Spence 1977), seek fast early growth to build market power (Spence 1979; Wernerfelt 1985), and invest heavily at early stages to move fast along the learning curve and gain a competitive edge through cost efficiency (Spence 1981). The rise and decline in the aggregate number of firms subject to these forces then describes the life cycle of the whole industry (Jovanovic and MacDonald 1994; Klepper 1996). That younger firms focus on growth and mature firms focus on profitability has also been borne out in empirical findings, such that stock prices react more to unexpected sales growth and capital expenditures among young firms (Anthony and Ramesh 1992), and that operating, investing and financing cash flows shows patterns consistent with various other life-cycle-related economic indicators (Dickinson 2011). Another rationale for these growth and profitability trends is that firms learn about their cost type over time and exit their industry if the learned information is

⁶ Abel and Eberly model a standard Cobb-Douglas production function and constant price elasticity of demand. Nezlobin et al. employ a classic production capacity framework that dates back to Arrow (1964). Further examples include Rogerson (2008, 2011), Nezlobin (2012), and McNichols, Rajan and Reichelstein (2014). In these studies, firms make overlapping capacity investments, the cumulative effect of which is the productive capacity that is available for producing output in any given period.

⁷ See also Rink and Swan (1979) for a survey.

unfavorable (Jovanovic 1982).⁸ The life cycle model in this paper abstracts from industry-level forces such as competition but is consistent with prior research in that it implies declining growth rates and increasing profitability as the firm matures. This firm-level life cycle emerges from the aggregation of individual investment (or product) life cycles that are assumed to start at increasingly advanced stages as the firm moves along its learning curve over time.

2. Model Structure and Notation

Consider a continuous-time setting in which a firm makes investments and subsequently collects the associated payoffs. An investment is defined as a commitment of resources that generates future cash inflows, where the term resource refers broadly to any productive means deployed at the firm's discretion.⁹ The amounts spent on investing will be referred to as expenditures, and the payoffs received from the investments will be referred to as revenues. For simplicity and ease of exposition, all resource flows are assumed to be settled in cash immediately, so that investment expenditures and revenues correspond to cash outflows and inflows, respectively. An extension to accrual-basis accounting is given in section 3.2. During the initial part of the analysis, the model is stated in terms of a deterministic evolution of investment opportunities. Uncertainty in the form of a stochastic state variable is added in section 3.3. The firm is assumed to have access to frictionless capital markets. Any excess of revenue over concurrent expenditures is paid out as dividends and, in the converse case, any shortfall is covered by raising additional capital.

The central objective is the derivation of an investment-level valuation function, hereafter denoted by v , with the desired property that, for any date t in the firm's life, v represents the present value of the future revenues generated by the investment expenditure made at that date. The value of v thus depends on both the amount of investment expenditure, hereafter denoted by k , and on the investment date t , which allows for changes in investment opportunities over time. In addition, it will be useful to let v compute the investment value not only as of its initiation date but also at all subsequent dates, after some of its payoffs have already been realized. To this end, v will also depend on the age of the investment, denoted by h .¹⁰ For a given h , v gives the pre-

⁸ Pastor and Veronesi (2003) likewise model learning in the sense of type-discovery, albeit in the form of firms' learning about their profitability over time and without entry/exit or other decision implications.

⁹ Depending on the business model, investing may, for example, correspond to purchasing a machine, acquiring a software license, hiring an employee, or paying a marketing agency to design an advertising campaign.

¹⁰ In standard neoclassical models, time-dependence of the investment problem is either unidimensional or specifies individual investments' output schedules independently of the evolution of the investment opportunity set. See

sent value, as of time t , of the future revenues left to be received after time $t + h$.¹¹ An investment value is thus completely described by the function $v(k; t, h)$.¹² The present value reflected in v is calculated by applying a discount rate r , which is assumed to be time-invariant and thus reflects the firm's long-term cost of capital. For convenience, v is assumed to be continuously differentiable in t and h , with $t, h \in \mathbb{R}^+$. Investment values are required to be finite in the sense that $v(k; t, h) \rightarrow 0$ as $h \rightarrow \infty$ for any t and k .

The standard neoclassical approach would at this point introduce a production technology that rationalizes the expenditures k that generate v , but, for the purpose of developing a testable valuation model, sans agency conflicts and other frictions, it suffices to evaluate past and future v along the path of first-best expenditures the firm actually undertakes, given by the solution to

$$\max_k (v(k; t, 0) - k) \quad (1)$$

for all t . There exist arbitrarily many technologies than can rationalize a given solution to (1), and they are empirically indistinguishable because deviations from the optimal path are, by definition, not observed. The analysis will therefore take the most parsimonious approach and assume that a unique investment plan exists that solves (1) at all t , and that this investment plan yields a continuously differentiable solution function $k(t)$.¹³ When evaluated along the path $k(t)$, the optimized investment value function can now be written as

$$v(t, h) \equiv v\left(k: k = \arg \max_k v(k; t, 0); t, h\right) \quad (2)$$

The discussion will return to the issue of identifying the associated investment plan $k(t)$ in section 3.1 and, for now, take its existence as given.

Given v and k , intrinsic firm value can be expressed as the sum of two components: the present value of the remaining future revenues from past investments, hereafter denoted by \bar{m} , and

Strebulaev and Whited (2011) for a review of the dynamic investment model. Berk, Green and Naik (1999) model an arrival process of investment opportunities in a real options framework. For an example of a capacity model with time evolution given by a market demand function, see Nezlobin, Rajan and Reichelstein (2014).

¹¹ For $h = 0$, one thus obtains the total present value of all future revenues from the investment; for $h = 1$, one obtains the present value of the future revenues received after date $t + 1$; etc.

¹² Implicit in this setup is the convenient but, for the purpose of this paper, not critical assumption that current investment opportunities are independent of the firm's past and future investment decisions, i.e., that $v(k; t, 0)$ does not depend on the firm's choice of k at dates before or after t . See Strebulaev and Whited (2011) for a discussion of dynamic models with interdependent investment choices.

¹³ For a general discussion of sufficient conditions for unique solutions to dynamic investment problems, see Stokey and Lucas (1989).

the net present value of all investments to be undertaken in the future, hereafter denoted by \vec{m} .

The past investment component, as of date t , is given by

$$\hat{m}(t) = \int_0^t e^{r(t-u)} v(u, t-u) du \quad (3)$$

where past expenditures k play no role because these are sunk costs.¹⁴ The net present value of future investments is given by

$$\vec{m}(t) = \int_t^\infty e^{r(t-u)} (v(u, 0) - k(u)) du \quad (4)$$

and so total intrinsic value can be written as $m = \hat{m} + \vec{m}$. A further variable of interest to the following discussion is the total revenue collected at any given time, hereafter denoted by w . The revenue (cash) flow at time $t+h$ from an investment made at time t is equal to $-e^{rh} v_h(t, h)$, and so total revenue flow, in aggregate across all investments, is

$$w(t) = - \int_0^t e^{r(t-u)} v_h(u, t-u) du \quad (5)$$

at any given time t .¹⁵ One can readily verify from (3) through (5) that the instantaneous return on investment in the firm is

$$w - k + m_t = rm$$

where $w - k$ is the net capital flow and m_t is the time-value change. The next section of the paper applies life cycle theory to develop concrete functional forms for v and k and provides a discussion of the resulting properties of firm value, revenue, growth rates and various ratios commonly studied in financial statement analysis. Section 4 shows some empirical evidence on how these predicted properties align with actual data.

3. Model Development and Analysis

3.1. Firm Value with Deterministic Investments

¹⁴ The firm's founding date in (3) has been normalized to zero, so that t indicates the firm's age.

¹⁵ To see why, let $f(t, h)$ denote the revenue flow at time $t+h$. By construction, $v(t, h)$ is the present value, as of time t , of all revenue to be received after time $t+h$, and so

$$v(t, h) = \int_{t+h}^\infty e^{r(t-u)} f(t, u-t) du$$

for all h . Differentiating with respect to h and solving for f yields

$$f(t, h) = -e^{rh} v_h(t, h)$$

where, as throughout this text, subscripts denote partial derivatives. Note that, by definition, v must be decreasing monotonically at the rate of the revenue flow, which implies $v_h < 0$ everywhere and thus $f > 0$.

Modeling the firm's learning curve requires addressing the question of how the payoffs from today's investment relate to the payoffs from yesterday's (last week's, last month's) investment, or, more formally, what the relationship is between the investment at date t and the firm's prior investments at dates $t - u$, for any $u \in (0, t)$. It will be useful to operationalize this relationship by asking specifically at which time $h(u)$ the value $v(t - u, h(u))$ of the investment made at time $t - u$ would equal the initial value $v(t, 0)$ of the investment made at time t . Now, if a firm advances along a learning curve as it matures and if competition intensifies as its industry matures, one should expect successive investments to begin at increasingly more advanced stages than their predecessors. For example, new products may be based on some previously developed platform, firms may learn from mistakes on prior projects and skip over trial-and-error phases, satisfied previous customers may be more likely to buy the firm's products again and thus accelerate the sales growth phase, and successful products may tend to draw additional competitors into the market and thus shorten the peak sales phase. Accordingly, one should think of $h(u)$ as an increasing function. In particular, consider the following linear representation.

Assumption 1 (Learning Curve). For any t and $u \in (0, t)$,

$$v(t, 0) = e^{ru}v(t - u, \theta u)$$

where $\theta \in \mathbb{R}^+$.

Assumption 1 ties the life cycle of individual investments to the evolution of the firm's investment opportunity set over time. Equating the initial value $v(t, 0)$ of a current investment to the later-stage values $v(t - u, \theta u)$ of prior investments implies that the new investment starts at a more advanced phase than its predecessors.¹⁶ This head start increases linearly at a rate of θ in the time distance u between the investment dates.¹⁷ (Recall that the second argument in $v(t, h)$ is the investment age h .) The real-life analogue to this assumption is that firms in their startup phase make investments with a long-term focus, e.g., in brand value, market share, reputation, or technological assets. Later investments benefit from the experience gained on earlier projects, so

¹⁶ The discount factor e^{ru} serves to make $v(t, \cdot)$ and $v(t - u, \cdot)$ comparable in present value terms. Recall that v is defined as a present value as of the investment date. Hence, prefixing $v(t - u, \cdot)$ by e^{ru} makes both sides of the equality present values as of time t .

¹⁷ Linearity is not a necessary condition for any of the results to follow, but tractability would suffer if one assumed a more complex relationship.

that mature firms with established reputation, customer base, market share and distribution networks are quicker to convert their efforts into cash and also likely face a more saturated market and a more competitive environment that curtail the lifespan of new products.¹⁸ Assumption 1 implies the following result. All proofs can be found in the Appendix.

Proposition 1. *The value at time $t + h$ of an investment made at time t is*

$$v(t, h) = e^{rt} f(\theta t + h)$$

for some function f such that $f'(\theta t + h) < 0$ everywhere and $\lim_{h \rightarrow \infty} f(\theta t + h) = 0$ for all t .

Proposition 1 is a statement about how quickly firms mature, but in a relative sense: the higher θ , the faster the value of new investment opportunities declines over time relative to the revenue flow from existing investments, and thus the shorter the life cycle of the firm as a whole relative to the life cycle of each individual investment. In the watershed scenario $\theta = 1$, each new investment scales up the firm's portfolio of investments at its status quo, so that the life cycle of the firm coincides with the life cycle of its first investment. At $\theta > 1$ ($\theta < 1$), the firm as a whole thus ages faster (more slowly) than its investments. In the extreme case $\theta = 0$, every investment is identical to its predecessors, so the firm never ages because it continues to launch equally profitable projects ad infinitum.¹⁹ Firms with low θ can therefore likely be found in stable industries with little technological progress and low competitive pressure. Conversely, if θ is high, profitable investment opportunities dissipate quickly and early in the firm's life, a scenario that may be characteristic of highly innovative and competitive environments. Graphically, v is compressed along the timeline as θ increases, whereas lowering θ produces a time dilation effect.

The linear learning curve describes how fast investment life cycles contract over time but not what the shape of the individual investments' life cycles looks like in the first place. However, Proposition 1 has reduced the problem of finding a general bivariate value function $v(t, h)$ to the equivalent of an initial value condition: one now immediately obtains a complete characteriza-

¹⁸ Assumption 1 implies that, in present value terms, the investments contributing most to future revenue are made during the firm's startup phase. One may think of early efforts to develop intellectual property, name recognition, or organizational culture that then pay off richly when the firm is mature. (Realizing these benefits may still require later investments to occur, but interdependence between investments is not critical to the model since the firm executes a unique optimal investment plan.) Also note that Assumption 1 makes no statement about profitability and therefore does not imply that startup-phase investments have the highest *net* present value.

¹⁹ Recall that the leading coefficient e^{rt} in v only serves to adjust for the time value effect, so that, at $\theta = 0$, investments made at different dates t are indeed identical in present value terms.

tion of *all* revenues throughout the firm's lifespan if one specifies the univariate function f in Proposition 1 to describe the revenue flow from just *one* investment, say, the startup one at $t = 0$. In the interest of developing a testable model, discussion will proceed with a parametric version of f , although it should be noted that a number of the results presented hereafter can be obtained under more general conditions.²⁰ The logistic function, stated in Assumption 2 below, offers a parsimonious but flexible representation of the introduction-growth-maturity-decline pattern commonly associated with the product life cycle idea.²¹

Assumption 2 (Product Life Cycle). *At time h , an investment made at date $t = 0$ has a value of*

$$v(0, h) = \frac{\beta_0}{1 + e^{\beta_1 + \beta_2 h}}.$$

One can readily verify that, as required, $v(0, h)$ is monotonically decreasing in h and approaches zero as $h \rightarrow \infty$ for any $\beta_0, \beta_2 > 0$. The parameters have natural economic interpretations: β_0 determines the total amount of payoff the investment yields over its lifetime, β_1 defines when the revenue flow reaches its peak, and β_2 sets the speed of revenue growth and decay.²² Similar to the effect of θ , a higher β_2 describes a more short-lived investment with revenues that rise and fall quickly. Using the result in Proposition 1 yields the general solution

$$v(t, h) = \frac{\beta_0 e^{rt}}{1 + e^{\beta_1 + \beta_2(\theta t + h)}} \quad (6)$$

for any t and h .²³ Figure 1 shows a graphical illustration of (6) and the associated revenue flow.

The final task to complete the model is determining the expenditure k that was required to produce the investment value v to begin with. To this end, consider the following rationale. Whenever firm A makes an investment by purchasing resources from firm B, the transaction amount becomes part of firm B's revenue. The investment cost of any firm must therefore map

²⁰ In particular, a number of results presented hereafter obtain generally for all logarithmically concave functions f .

²¹ The logistic curve is the solution to the logistic differential equation $f(x)(1 - f(x)) = f'(x)$ and is commonly used in scientific models of, for example, diffusion processes or population growth.

²² For firms with multiple distinct business segments, operating divisions, geographic locations, subsidiaries, or product lines, one can also specify a v -function for each such subunit.

²³ The steps to obtain equation (6) are

$$v(0, h) = f(h) = \frac{\beta_0}{1 + e^{\beta_1 + \beta_2 h}}$$

and so

$$f(\theta t + h) = \frac{\beta_0}{1 + e^{\beta_1 + \beta_2(\theta t + h)}}.$$

into the revenue functions of the firm's suppliers. In particular, if one were to bundle all of firm A's expenditures, one would obtain the revenue curve of a (hypothetical) single-source supplier of firm A. Since the solution in Proposition 1 applies generically to *any* firm, this hypothetical supplier's revenue curve would also be given by a revenue function w , as specified in (5), albeit with parameter values different from those in firm A's own revenue function and with a founding date preceding that of firm A. Assumption 3 below formalizes the idea.

Assumption 3. *The investment expenditure function k has the same functional form as the revenue w of a firm with a founding date of $-\infty$. Any firm's investment at $t = 0$ earns a net present value of zero, and the internal rate of return on investments is finite at all $t > 0$.*

A net present value of zero from the investment made at the firm's founding date $t = 0$ essentially means that entrepreneurs are rational. If the investment at $t = 0$ were strictly profitable, then, by continuity, there must exist investment opportunities at just slightly earlier dates $t < 0$ that also have positive net present value. A profit-maximizing entrepreneur would therefore have chosen a founding date $t < 0$.²⁴ The assumption of a finite rate of return is made to ensure that revenues can never be generated 'out of thin air' but require a non-zero level of resource expenditure. Setting the hypothetical supplier's founding date at $t = -\infty$ is done for parsimony but does not critically alter the behavior of k . One now obtains the following expenditure function.

Proposition 2. *The firm's investment expenditure is given by*

$$k(t) = \frac{\beta_0 e^{rt}(1 + e^\alpha)}{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta t})}$$

where all parameter values except α are the same as in the associated investment value v .

For any time t in the firm's life, the two components of firm value in (3) and (4) can now be evaluated explicitly as

$$\bar{m}(t) = \int_0^t e^{r(t-u)} v(u, t-u) du = \frac{\beta_0 e^{rt}}{\beta_2(1-\theta)} \ln \left(\frac{1 + e^{-\beta_1 - \beta_2 \theta t}}{1 + e^{-\beta_1 - \beta_2 t}} \right) \quad (7)$$

for the present value of the remaining future revenues from the firm's past investments, and

²⁴ This observation is also consistent with the conclusions by Wernerfelt (1985).

$$\begin{aligned}
\bar{m}(t) &= \int_t^\infty e^{r(t-u)}(v(u, 0) - k(u)) du \\
&= \frac{\beta_0 e^{rt}}{\beta_2 \theta} \left(\ln(1 + e^{-\beta_1 - \beta_2 \theta t}) - \frac{1 + e^\alpha}{1 + e^{\beta_1}} \ln(1 + e^{-\alpha - \beta_2 \theta t}) \right)
\end{aligned} \tag{8}$$

for the net present value of all remaining future investments. Consistent with the conventional wisdom that the intrinsic firm value m must equal the aggregation of discounted future net cash flows over the remaining life of the firm, one can verify by straightforward algebra that the sum of the (net) investment values in (7) and (8) is indeed equivalent to

$$m(t) = \int_t^\infty e^{r(t-u)}(w(u) - k(u)) du \tag{9}$$

where w is the firm's total revenue from (5). Proposition 3 below gives a formal summary.

Proposition 3. *At any given time t , intrinsic firm value is equal to*

$$\begin{aligned}
m(t) &= \frac{\beta_0 e^{rt}}{\beta_2 \theta (1 - \theta)} \left(\ln(1 + e^{-\beta_1 - \beta_2 \theta t}) - \theta \ln(1 + e^{-\beta_1 - \beta_2 t}) \right. \\
&\quad \left. - \frac{(1 - \theta)(1 + e^\alpha)}{1 + e^{\beta_1}} \ln(1 + e^{-\alpha - \beta_2 \theta t}) \right).
\end{aligned}$$

3.2. Growth Rates and Valuation Ratios

Standard valuation practice typically relies on a steady-state assumption to describe long-term cash flows, which tends to produce value estimates dominated by a large terminal value component. Consider instead the time-series behavior of the investment value in (6). Consistent with the intuition that firms move along a learning curve and face increased competition over time, the revenue from investments made later in the firm's life is realized faster than the revenue from investments made during the firm's startup phase. In particular, the percentage

$$\frac{v(t, h)}{v(t, 0)} = \frac{1 + e^{\beta_1 + \beta_2 \theta t}}{1 + e^{\beta_1 + \beta_2 (\theta t + h)}} \tag{10}$$

of the revenue of an individual investment made at time t that, in present value terms, has not yet been collected h periods later decreases monotonically in t for any given h . Similar acceleration occurs at the level of firm's investment opportunity set, where the ratio

$$\frac{e^{-ru} v(t + u, 0)}{v(t, 0)} = \frac{1 + e^{\beta_1 + \beta_2 \theta t}}{1 + e^{\beta_1 + \beta_2 \theta (t + u)}} \tag{11}$$

of the value of the investment at time t relative to the value of the investment u periods later likewise decreases in t .

Both (10) and (11) are decreasing in β_2 and θ , but whereas β_2 sets the speed of decay at both the firm-level and the investment-level *jointly*, θ determines the length of individual investments' life cycles *relative* to the life cycle of the firm's investment opportunity set. To see this, note that the ratio of (10) to (11) is monotonically increasing in θ , i.e., the decline in investment opportunities outpaces the contraction of the individual investments' revenue cycles as θ increases, as noted in the context of Proposition 1 above.²⁵ By contrast, the directional effect of β_2 on this ratio varies depending on the value of θ .

The life cycle model provides a theoretically motivated alternative to the constant growth rate assumption commonly made as a practical but ad-hoc expedient in standard valuation models. In particular, investment values in the life cycle model grow at a time-dependent rate of

$$\frac{v_t(t, 0)}{v(t, 0)} = r - \frac{\beta_2 \theta}{1 + e^{-\beta_1 - \beta_2 \theta t}} \quad (12)$$

that is monotonically declining in t , i.e., young companies grow faster than mature firms. For $r < \beta_2 \theta$, the growth rate becomes negative in the long term. The firm eventually enters into decline and investment opportunities vanish as $t \rightarrow \infty$. In the converse case $r > \beta_2 \theta$, the firm continues to grow indefinitely.²⁶ In the knife-edge case $r = \beta_2 \theta$, investment opportunities settle at a long-term steady state level of $\beta_0 e^{-\beta_1}$. Equation (12) thus provides an explicit mapping between growth and the cost of capital, which arises because the life cycle model makes a connection between successive generations of investments. This connection not only implies a growth trajectory but at the same time requires adjustments for the time value of money between investment dates. In a standard constant growth rate model, by contrast, the only connection is that growth must be bounded above by r since otherwise firm value would become infinite.²⁷

²⁵ Note, however, that *ceteris paribus* parameter changes do not necessarily predict testable outcomes because the parameters of v are themselves determined jointly in an unobservable equilibrium of decisions and expectations by the firm's management, investors, customers and suppliers. For example, if the persistence in a firm's stream of investment opportunities is associated with the lifespan of its products, one should expect to see correlation between θ and β_2 .

²⁶ A natural restriction common in practice is to require the asymptotic growth rate $r - \beta_2 \theta$ to lie below the long-term growth rate of the general economy. The empirical test in section 4 returns to this point. In making these comparisons, one should also bear in mind that r , θ and β_2 are likely to be determined jointly in reality.

²⁷ The growth rate in (12) is also bounded above by r , as a consequence of the linear relationship between t and h implied by Assumption 1. More analytically complex versions of Assumption 1, however, can produce growth rates that temporarily exceed r .

The investment value v is unobservable, and hence (12) cannot be computed directly from firm's reported financial information. One may therefore ask to what extent the growth behavior of revenue (which is observable) mirrors that of v . The revenue w produced at time t by all investments the firm has made until that time obtains after substitution of (6) into (5), which yields

$$\begin{aligned} w(t) &= - \int_0^t e^{r(t-u)} v_h(u, t-u) du = \frac{1}{1-\theta} (v(t,0) - e^{rt}v(0,t)) \\ &= \frac{\beta_0 e^{rt}}{1-\theta} \left(\frac{1}{1+e^{\beta_1+\beta_2\theta t}} - \frac{1}{1+e^{\beta_1+\beta_2 t}} \right) \end{aligned} \quad (13)$$

for any $t \geq 0$.²⁸ Revenue therefore grows at a rate of

$$\frac{w'(t)}{w(t)} = r - \frac{\beta_2}{e^{\beta_1+\beta_2 t} - e^{\beta_1+\beta_2\theta t}} \left(\frac{1+e^{\beta_1+\beta_2 t}}{1+e^{-\beta_1-\beta_2\theta t}} \theta - \frac{1+e^{\beta_1+\beta_2\theta t}}{1+e^{-\beta_1-\beta_2 t}} \right) \quad (14)$$

which, like v , is monotonically decreasing over time but, in contrast to v , is not bounded above by the firm's cost of capital.²⁹ In further contrast to v , neither (13) nor (14) is always monotonic in θ and β_2 across all t , i.e., the impact of a more condensed or protracted life cycle on revenue levels and growth can vary depending on the firm's age. As $t \rightarrow \infty$, revenue declines toward zero if $r < \min\{\beta_2, \beta_2\theta\}$ and diverges in the converse case. Since w begins at zero at the firm's founding date $t = 0$, the revenue of firms with $r < \min\{\beta_2, \beta_2\theta\}$ therefore reaches a unique interior maximum at some point during the firm's lifespan.³⁰ Recalling that the analogous separating condition for the investment value v is $r < \beta_2\theta$, one thus finds that revenue and investment value need not evolve comonotonically.³¹ In particular, the ratio of revenue to the value of contemporaneous new investments is given by

$$\frac{w(t)}{v(t,0)} = \frac{1}{1-\theta} \left(1 - \frac{1+e^{\beta_1+\beta_2\theta t}}{1+e^{\beta_1+\beta_2 t}} \right) \quad (15)$$

²⁸ In the special case $\theta = 1$, the revenue function evaluates to

$$w(t) = \frac{\beta_0 \beta_2 t e^{rt}}{(1+e^{\beta_1+\beta_2 t})(1+e^{-\beta_1-\beta_2 t})}$$

in view of L'Hôpital's rule.

²⁹ See the proof of Corollary 1 below for further detail.

³⁰ The separating case $r = \min\{\beta_2, \beta_2\theta\}$ leads to a stable long-run revenue level of $\beta_0 e^{-\beta_1}/|1-\theta|$. In the special case $\theta = 1$, $r = \beta_2 = \beta_2\theta$ and thus revenue diverges to ∞ as $t \rightarrow \infty$.

³¹ When $\theta > 1$, $r \in (\beta_2, \beta_2\theta)$ implies that investment opportunities $v(t,0)$ decline to zero in the long term while revenues continue to grow. The scenario $r > \beta_2$ also implies that the (undiscounted) payoffs from the firm's investments grow indefinitely. This outcome may appear counterintuitive, and the empirical results in section 4 indeed suggest that firms with $\theta > 1$ and $r > \beta_2$ are rare.

which is monotonically increasing in t and either, in case of $\theta < 1$, reaches a long-term stable value of $\frac{1}{1-\theta}$ or, in case of $\theta > 1$, diverges to ∞ as $t \rightarrow \infty$.³² There exists a time T , implicitly characterized as the unique solution to

$$\frac{1 + e^{\beta_1 + \beta_2 \theta T}}{1 + e^{\beta_1 + \beta_2 T}} = \theta$$

at which $w(T) = v(T, 0)$, i.e., a time when the firm's revenue matches the value of its contemporaneous new investment exactly.³³ Revenues understate new investment values during the early stages of the firm's life, at $t < T$, and exceed the value of new investments at $t > T$ when the firm is mature. The preceding observations about growth are formally summarized below.

Corollary 1. *Regarding growth in revenue vis-à-vis investment values, the model implies that*

- *the rates of growth in revenue and in the value of new investments both decrease monotonically as the firm ages;*
- *the growth rate in the value of new investments is bounded above by the firm's cost of capital, but the revenue growth rate is unbounded; and*
- *the ratio of revenue to the value of concurrent new investments, w/v , increases monotonically in firm age t , and it is zero at $t = 0$ and converges to a long-term value of*

$$\lim_{t \rightarrow \infty} \frac{w(t)}{v(t)} = \frac{1}{1 - \theta}$$

if $\theta < 1$ and diverges to ∞ if $\theta \geq 1$.

The ratio of the value of a marginal unit of capital to its cost has been studied extensively in the form of Tobin's q . In the context of the life cycle model, marginal Tobin's q is computed as

$$\frac{v(t)}{k(t)} = \frac{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta t})}{(1 + e^{\alpha})(1 + e^{\beta_1 + \beta_2 \theta t})} \quad (16)$$

³² In the special case $\theta = 1$, (10) becomes

$$\frac{w(t)}{v(t, 0)} = \frac{\beta_2 t}{1 + e^{-\beta_1 - \beta_2 t}}$$

³³ When $\theta = 1$, one obtains the exact solution

$$\beta_2 T = 1 + e^{-\beta_1 - \beta_2 T} \Leftrightarrow T = \frac{1 + z(e^{-\beta_1 - 1})}{\beta_2}$$

where $z(\cdot)$ denotes the product logarithm.

Inspection of (16) shows that marginal q is monotonically increasing in t , i.e., the investments of a mature firm earn a higher return than the investments made during a firm's startup phase. The ratio is monotonically increasing in α , and the minimum level $\alpha = \beta_1$ implies $k(t) = v(t, 0)$ at all t and thus describes a firm operating in a perfectly competitive environment with zero net present value investments only.³⁴ Marginal q and the firm's growth trajectory jointly determine intrinsic value: the former defines how much value each investment creates, and the latter determines how fast the investment opportunities run out. Based on the well-established notion in economics that high profits cannot be sustained in the long run because of competitive pressure, one might posit that firms with high marginal q should be more short-lived than firms with low marginal q . In parametric form, one would thus hypothesize a positive correlation between α and $\beta_2\theta$. (Recall that, the higher β_2 and θ , the faster the firm moves through its life cycle.)

The marginal q ratio in (16) is not observable in practice, but investors do have ready access to information about revenue and expenditure.³⁵ The initial values $w(0) = 0$ and $k(0) > 0$ imply that firms, as is common in reality, begin their career with a negative net cash flow, i.e., $w < k$. The break-even point is reached when

$$w(t) = k(t) \Leftrightarrow \frac{1 + e^{\alpha + \beta_2\theta t}}{1 + e^{\beta_1 + \beta_2\theta t}} - \frac{1 + e^{\alpha + \beta_2\theta t}}{1 + e^{\beta_1 + \beta_2 t}} = \frac{(1 - \theta)(1 + e^\alpha)}{1 + e^{\beta_1}}$$

which holds at a unique time $t = T$, so that every firm's life cycle can be divided into a startup phase $(0, T)$ during which the firm consumes more cash than it generates, and a maturity phase (T, ∞) during which net cash flow is positive. More profitable firms and firms with a more condensed life cycle start generating positive net cash flow earlier, i.e., T is decreasing in α and in $\beta_2\theta$. The ratio of w to k , which corresponds to a gross return ratio under simple cash-basis accounting, is monotonically increasing over time and, as $t \rightarrow \infty$, converges to a finite value above unity if $\theta < 1$ and diverges to ∞ if $\theta \geq 1$. By the properties of w/v identified in Corollary 1, the cash return ratio lies below marginal Tobin's q early in the firm's life but rises monotonically to exceed marginal q later. The result below summarizes these observations.

Corollary 2. *Regarding the revenue and expenditure cash flows w and k , the model implies that*

³⁴ The analysis presented here assumes that $\alpha \geq \beta_1$. At $\alpha < \beta_1$, all investments have a negative net present value, a scenario incompatible with the model's assumption that firms invest optimally.

³⁵ These would be reported as operating and investing cash flows. Since the firm is operating in an efficient capital market environment, its financing cash flow is a dividend in the amount of $w - k$ when $w > k$ and a capital inflow of $k - w$ (e.g., a stock or bond issuance) when $w < k$.

- there exists a unique firm age T such that $w(t) \leq k(t)$ if and only if $t \leq T$, and the value of T decreases in α and in θ ;
- the cash-basis gross return ratio w/k increases monotonically in firm age t , and this ratio is zero at $t = 0$ and converges to a long term value of

$$\lim_{t \rightarrow \infty} \frac{w(t)}{k(t)} = \frac{1 + e^{-\beta_1}}{(1 - \theta)(1 + e^{-\alpha})}$$

if $\theta < 1$ and diverges to ∞ if $\theta \geq 1$; and

- the cash basis gross return ratio increases monotonically in t relative to marginal Tobin's q , and there exists a unique point in time in the firm's life at which the two ratios are equal.

A common heuristic in practice is to value firms via multiples of their reported sales revenue. In the life cycle model, the ratio of the firm's intrinsic value m to its revenue w , hereafter referred to as the market-to-sales ratio, is

$$\frac{m(t)}{w(t)} = \frac{\ln(1 + e^{-\beta_1 - \beta_2 \theta t}) - \theta \ln(1 + e^{-\beta_1 - \beta_2 t}) - \frac{(1 - \theta)(1 + e^{-\alpha})}{1 + e^{\beta_1}} \ln(1 + e^{-\alpha - \beta_2 \theta t})}{\beta_2 \theta \left(\frac{1}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{1}{1 + e^{\beta_1 + \beta_2 t}} \right)} \quad (17)$$

for any $t \geq 0$.³⁶ It stands to reason that (17) exhibits predictable regularities, as the asymptotic behaviors of w and m are similar: like revenue, firm value diverges as $t \rightarrow \infty$ when $r > \min\{\beta_2, \beta_2 \theta\}$ and declines to zero when $r < \min\{\beta_2, \beta_2 \theta\}$, while the knife-edge case $r = \min\{\beta_2, \beta_2 \theta\}$ delivers a steady state value of

$$\lim_{t \rightarrow \infty} m(t) = \frac{\beta_0 e^{-\beta_1}}{r} \max \left\{ \frac{1}{\theta - 1}, \frac{1}{1 - \theta} - \frac{1 + e^{-\alpha}}{1 + e^{-\beta_1}} \right\}.$$

The following results summarizes two important properties.

Corollary 3. *The market-to-sales ratio m/w is monotonically decreasing in firm age and approaches a long-term value of*

$$\lim_{t \rightarrow \infty} \frac{m(t)}{w(t)} = \frac{1}{\beta_2} \max \left\{ 1, \frac{1}{\theta} - \frac{(1 - \theta)(1 + e^{-\alpha})}{\theta(1 + e^{-\beta_1})} \right\}.$$

³⁶ Technically, w measures revenue on a cash basis, whereas sales multiples in practice tend to use accrual-basis revenue, but the two are easily commutable through changes in receivables and accrued and deferred revenue.

The decreasing market-to-sales ratio obtains because firms in their startup phase build up revenue faster than they build up intrinsic value, whereas mature firms convert their intrinsic value into revenue at an increasingly fast pace. To understand why, note first that, by Corollary 1, the revenue growth rate decreases over time and therefore consistently outpaces the growth rate in intrinsic value among young firms, whose intrinsic value always contains the more slowly growing future revenues. In mature, declining firms, current revenue becomes increasingly large relative to the remaining future revenues. Second, revenue increases relative to expenditure as the firm matures, by Corollary 2, and so the additions to firm value from new investments decrease continually relative to the reductions in firm value from revenue realizations.

The time-dependence of the revenue multiple is noteworthy not only because valuation practice tends to focus more on a firm's industry membership or peer group, rather than its age, as the critical determinant of the benchmark ratio, but also because revenue multiples are most widely applied to (not yet profitable) firms in their startup phase, when the market-to-sales relation changes fastest.³⁷ Another common heuristic to be cautioned against is the assignment of higher multiples to firms with higher growth. Although the limit value in Corollary 3 suggests that firms with a more protracted life cycle (lower θ or β_2) have a higher long-term market-to-sales ratio, growth and life cycle duration are not monotonically related, as one can verify by observing that the impact of changes in θ and β_2 on the revenue growth rate in (14) can vary across t .³⁸

A number of the most widely used financial metrics, including the market-to-book ratio and the return on assets, require accrual-basis earnings and book (balance sheet) values. Two critical features of accrual accounting are the capitalization of expenditures on the balance sheet at their original cost (the historical cost principle) and their subsequent expensing in proportion to the revenues they generate (the matching principle). If one applied these rules, the balance sheet value at time $u \geq t$ of an investment made at time t should equal the investment cost $k(t)$ multiplied by the percentage of the investment's total revenue that has not been collected yet. The following assumption formalizes this principle.³⁹

³⁷ That the market-to-sales ratio decreases most around the founding date follows from the observation that $m(0)/w(0) = \infty$ for any $\alpha > \beta_1$.

³⁸ In fact, straightforward calculation on (14) shows that, for any $\beta_1 < 0$, startup firms' revenue growth rate increases in β_2 while its market-to-sales ratio decreases.

³⁹ The accrual model presented here is focused on expenses and assumes that accrual-basis revenues are the same as cash receipts. In practice, short-term differences can arise between the latter two when accounts receivable and accrued and deferred revenue balances change.

Assumption 4. For all t and $u \geq t$, the balance sheet carrying amount at time u of the expenditure made at time t is $k(t) \frac{v(t,u-t)}{v(t,0)}$.

Figure 2 illustrates Assumption 4 by numerical example. In interpreting the graph, recall that expenditures include all costs incurred by the firm, such as labor, supplies, and capital assets, and that these vary in how they contribute to future revenues. The figure shows the possible confluence of these dynamics: a high rate of expensing at the early stages, dominated by labor and other costs with a short horizon of benefit (some of which may be temporarily capitalized in inventory and expensed with some delay), and a lower expense at later dates when only the long-term capital assets remain, possibly including some that are not subject to scheduled expensing at all and held at their purchase cost indefinitely, such as land and certain intangibles. Assumption 4 is a parsimonious representation of accrual accounting in that it relies only on the economic primitives of the model without requiring additional parameters. Many of the idiosyncratic details of accounting practice are invariably omitted, but the model is readily amenable to extensions if needed, say, to accommodate more specific accounting conventions.⁴⁰

Aggregating the carrying amounts of all past investment expenditures yields the firm's (net) book (or balance sheet) value, hereafter denoted by

$$b(t) \equiv \int_0^t k(u) \frac{v(u, t-u)}{v(u, 0)} du \quad (18)$$

The need for balance sheet and income statement to articulate implies that accrual-basis earnings, hereafter denoted by p , and (net) dividends, $w - k$, must fully explain all changes in b , i.e.,

$$b_t(t) = p(t) - w(t) + k(t)$$

for all t , and thus

$$p(t) = w(t) + b_t(t) - k(t) = w(t) + \int_0^t k(u) \frac{v_h(u, t-u)}{v(u, 0)} du \quad (19)$$

In contrast to the net cash flow $w - k$, which is negative during the firm's startup phase, earnings are zero at the firm's founding date $t = 0$ and take non-negative values at all $t > 0$, provid-

⁴⁰ For example, one could permit some proportion of costs to be expensed as incurred, as is common, for example, in research and development efforts. The balance sheet and related ratios would be scaled as a result but their properties would otherwise be left unchanged. For further discussion of the effects of conservatism on the relationship between book value and firm value, see Feltham and Ohlson (1995), Zhang (2000) and Penman and Zhang (2002).

ed that $r \geq 0$ and $\alpha \geq \beta_1$.⁴¹ To avoid unproductive case distinctions, the subsequent discussion will assume that both of these conditions hold, since firms in reality are unlikely to have a negative cost of capital or to make investments with negative net present values.

The ratio of p to b is the firm's return on assets, a central accounting-based profitability ratio whose economic interpretation is distinct from the metrics examined so far. Unlike the cash-basis profit ratio w/k , which matches revenues from past investments with the cost of the current investment, and marginal Tobin's q , v/k , which measures the value creation of current investment only, the return on assets measures the current profitability of all past investments while matching related costs and revenues. Some key properties of p/b are summarized below.

Corollary 4. *The return on assets is increasing in firm age t , in the cost of capital r , and in the profitability parameter α . The return is zero at $t = 0$ and converges to a long-term value of*

$$\lim_{t \rightarrow \infty} \frac{p(t)}{b(t)} = \frac{(1 + e^{-\beta_1})r + \beta_2(1 - \theta)(e^{-\beta_1} - e^{-\alpha})}{(1 - \theta)(1 + e^{-\alpha})}$$

if $\theta < 1$ and of ∞ if $\theta \geq 1$.

A higher return on assets can arise from a combination of three factors: higher economic profitability, a shorter lifespan, and a higher cost of capital. The first two mirror the behavior of w/k and v/k , but the dependence on the cost of capital is unique to p/b and arises from the accrual principle of matching of historical investment costs to future revenues. To see why, consider scaling up both the future revenues and the discount rate of an investment in a manner such that the net present value of the investment remains constant. The expenses recognized at each point during the investment's life then remain the same, but p/b increases.

The increase of the return on assets in firm age t tracks the contemporaneous increase in marginal Tobin's q , v/k . Yet, the conjecture that profitability ratios generally move in parallel to marginal q is not correct. As the following results show, ratios involving either one of earnings or book value, but not both, can have multiple minima and maxima during the life of the firm. To illustrate the point, contrast Corollary 4 with two alternate metrics related to profitability, both of

⁴¹ The model can rationalize negative earnings by either of two simple changes. First, adding a stochastic component, as shown in the next section, can produce $p < 0$ if realized revenue is less than its prior expected value. Second, modifying Assumption 4 to prescribe that some part of the firm's investment cost be expensed as incurred can produce $p < 0$ during the firm's startup phase, while $k > w$.

which find widespread use in financial analysis: the return on sales, defined as the ratio of p to w , and the market-to-book ratio, defined as the ratio of m to b .

To make the properties of these ratios transparent, it will be convenient to define the firm's gross book value \bar{b} as the amount of accumulated, unamortized past investment expenditures, i.e.,

$$\bar{b}(t) = \int_0^t k(u) du \quad (20)$$

One can now decompose the firm's return on sales into the product

$$\frac{p(t)}{w(t)} = \frac{p(t)}{b(t)} \cdot \frac{b(t)}{\bar{b}(t)} \cdot \frac{\bar{b}(t)}{k(t)} \cdot \frac{k(t)}{w(t)} \quad (21)$$

of which the first and last component ratios on the right side are increasing and decreasing, respectively, in view of Corollaries 4 and 2.⁴² The ratio of gross book value to expenditure, \bar{b}/k , is an increasing function of firm age t because the growth rate in investment cost is decreasing over time. The ratio of net-to-gross book value, b/\bar{b} , is a decreasing function of t , in view of two observations. First, as time passes, the decreasing growth in expenditures implies that newer investments, which still have a high carrying value, make up a lesser proportion of total assets relative to older investments, which have a lower net carrying value.⁴³ Second, the rate of expensing, relative to an investment's carrying amount, increases over time, as one can verify by noting that

$$\frac{-v_h(t, h)}{v(t, h)} = \frac{\beta_2}{1 + e^{-\beta_1 - \beta_2(\theta t + h)}}$$

is increasing in h .

The inconsistent directionality of the four component ratios in (21) yields a time series that, unlike that of the return on assets, is generally not monotonic and may have multiple maxima and minima. Figure 3 shows the evolution of the return on sales, as a function of t , in three different scenarios, including a monotonic case, a single-peaked case, and a case with multiple extrema.⁴⁴ Non-monotonicity occurs even though return on sales and return on assets share a num-

⁴² In the standard language of financial analysis, the product of the last three component ratios on the right-hand side of (21) is the reciprocal of the firm's asset turnover.

⁴³ See Lemma 1 at the end of the appendix for a formal proof.

⁴⁴ The existence and location of the extrema depend on all relevant model parameters jointly, and there is no conveniently interpretable, closed-form analytical representation to distinguish the various scenarios possible.

ber of properties, including that both have an initial value of zero at $t = 0$ and increase in both r and α .⁴⁵ Corollary 5 below provides a summary.

Corollary 5. *The return on sales is increasing in the cost of capital r and in the profitability parameter α , but is not generally monotonic in firm age t . The return on sales is zero at $t = 0$ and approaches a long-term value of*

$$\lim_{t \rightarrow \infty} \frac{p(t)}{w(t)} = 1 - \max \left\{ 0, \frac{\beta_2(1 - \theta)(1 + e^{-\alpha})}{(\beta_2(1 - \theta) + r)(1 + e^{-\beta_1})} \right\}$$

The market-to-book ratio may, similarly, reach multiple stationary points over the course of the firm's life. This claim somewhat defies the intuition that book value, which accumulates ever more past investments as the firm matures, should increase over time relative to market value, which reflects the shrinking set of payoffs that remain to be reaped in the future. The reason for the failure of this intuition can again be understood by decomposing the ratio, in this case, into

$$\frac{m(t)}{b(t)} = \frac{m(t)}{w(t)} \cdot \frac{w(t)}{k(t)} \cdot \frac{k(t)}{\bar{b}(t)} \cdot \frac{\bar{b}(t)}{b(t)} \quad (22)$$

The properties of the market-to-sales ratio, m/w , on the right-hand side of (22) are familiar from Corollary 3, and the remaining three elements are the same as in (21), yielding four component ratios that are, in order, decreasing, increasing, decreasing and increasing in t . Figure 4 shows a graphical illustration of several possible scenarios, including cases of multiple local extrema. The next result summarizes some additional properties.

Corollary 6. *The market-to-book ratio is increasing in the cost of capital r and in the profitability parameter α , but is not generally monotonic in firm age t . At $t = 0$, the market-to-book ratio is ∞ if $\alpha > \beta_1$ and 1 if $\alpha = \beta_1$, and it approaches a long-term value of*

$$\lim_{t \rightarrow \infty} \frac{m(t)}{b(t)} = \left(1 + \frac{r}{\beta_2(1 - \theta)} \right) \left(1 + \frac{e^{-\beta_1} - e^{-\alpha}}{\theta(1 + e^{-\alpha})} \right)$$

if $\theta < 1$ and of ∞ if $\theta \geq 1$.

⁴⁵ A smaller point of difference between the two is that, as shown in Corollary 5 below, the return on sales, by construction, remains finite and bounded above by 1 at all t , whereas the return on assets may diverge as $t \rightarrow \infty$.

Corollary 6 makes a cautionary statement regarding two common interpretations of the market-to-book ratio. First, m/b is often read as an indicator of value creation in the sense of Tobin's q , which is indeed correct in that m/b is increasing in α . Yet, even in the extreme case $\alpha = \beta_1$, when every investment earns a net present value of zero, the market-to-book ratio does not collapse to 1 but, as the limit value shows, remains in excess of 1 by an amount that is increasing in the firm's discount rate r . Similar to the case of the return on assets, the reason is that past investment costs included in b are based on historical cost and are not adjusted for time value of money effects as time passes, whereas the intrinsic value in the numerator always reflects the present value of the future net cash flows as of the valuation date. The higher r , the more book value lags behind market value, even if the firm were to generate an economic profit of zero.⁴⁶

A second interpretation views a high market-to-book ratio as an indication of sizeable future investment opportunities, relative to the value of investments already in place. To see where this intuition can fail, consider an increase in θ , which implies slower growth (or faster decline) in investment opportunities and thus, as one can see from inspection of (7) and (8), indeed always reduces the ratio of future investment opportunity value, \bar{m} , to the value of investments in place, \bar{m} . Yet, the limit value of the market-to-book ratio in Corollary 6 is not monotonically decreasing in θ but u-shaped, i.e., a reduction in the future component of firm value can in fact be associated with an *increase* in the market-to-book ratio. The cause again lies in the historical cost principle. Decreasing growth in investment opportunities not only diminishes the future value component, and thus decreases m/b , but simultaneously increases the ratio between the market value and the historical cost-based book value of the investments in place, and thus increases m/b . The reason for the latter effect is that diminished growth gives greater weight to the older investments, whose market-to-book relation has been most inflated by historical cost accounting, and, by the same logic as discussed above, this effect becomes more pronounced the higher the discount rate r . The resulting increase in the market-to-book ratio therefore occurs precisely because the firm does *not* face abundant future opportunities.

⁴⁶ Similarly, McNichols et al. (2014) observe that the market-to-book ratio of firms with zero economic profit is increasing in the discount rate r for amortization policies that do not depend on r . A long-run market-to-book ratio of 1 is, on the other hand, theoretically obtainable if the expensing schedule recognizes the time value of money, e.g., by the relative benefit depreciation rule (Dutta and Reichelstein 2005). This may, however, require the recognition of negative expenses.

3.3. Stochastic Life Cycles

In reality, firms do not follow deterministic paths but are subjected to random events, and one may ask to what extent the results discussed thus far carry over to a world in which investment opportunities and payoffs are stochastic.⁴⁷ To integrate uncertainty into the model, consider v as a stochastic object and interpret the life cycle as a statement about the expected, rather than the realized, trajectory taken by successive investments. Assumption 1 can then be restated as

$$E(v(t, 0)) = E(e^{ru}v(t - u, \theta u))$$

for any expectation taken between times $t - u$ and t . As in the deterministic setup, the firm's expected future evolution is then characterized by investments made at increasingly advanced stages, but realized outcomes can deviate from this path.

To make the idea concrete and consistent with the parametric version developed in the preceding sections, interpret the scale parameter β_0 as a stochastic state variable that evolves according to the geometric Brownian motion

$$d\beta_0(t) = \sigma\beta_0(t) dB(t) \quad (23)$$

where $B(t)$ is a standard Wiener process. One then obtains the standard solution

$$\beta_0(t) = \beta_0(0)e^{\sigma B(t) - 0.5\sigma^2 t}$$

and $E(X(t + h)|X(t)) = X(t)$ for all t and h .⁴⁸ The initial condition in Assumption 2 can then be recast by setting $\beta_0 = \beta_0(0)$, so that the stochastic equivalent to deterministic investment value function becomes

$$v(t, h) = \beta_0(t + h) \frac{e^{rt}}{1 + e^{\beta_1 + \beta_2(\theta t + h)}} = \beta_0(0) \frac{e^{rt + \sigma B(t+h) - 0.5\sigma^2(t+h)}}{1 + e^{\beta_1 + \beta_2(\theta t + h)}} \quad (24)$$

Likewise, the investment cost function is now

$$k(t) = \beta_0(t) \frac{(1 + e^\alpha)e^{rt}}{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2\theta t})} = \beta_0(0) \frac{(1 + e^\alpha)e^{(r - 0.5\sigma^2)t + \sigma B(t)}}{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2\theta t})} \quad (25)$$

if one maintains the same assumptions as in the original model.⁴⁹ The results of the previous section carry over to this setup, albeit as properties that hold in expectation rather than with certain-

⁴⁷ Uncertainty may be relevant to explaining variation in model parameters across firms. For example, in a standard risk-based asset pricing framework, uncertainty is a prerequisite for explaining differences in the discount rate r across firms, as payoffs in a deterministic world should all be subject to the same risk-free rate.

⁴⁸ Many alternative specifications are, of course, possible. Importantly, however, since v is defined as a present value of expected future cash flows, $\beta_0(t)$ must be a martingale in order to maintain the identity

$$v(t, 0) = - \int_0^\infty E(e^{-ru}v_h(t, u)) du$$

ty. Importantly, this structure makes no assumptions about the correlation of state variables across firms. The model can therefore accommodate common notions of risk arising from the comovement of payoffs among firms and can thus be integrated into asset pricing theories built on risk-based differences in discount rates.

4. Empirical Test

This section takes the life cycle model to an empirical test, which is presented in two parts. First, an illustration is given how the parameters of the model can be estimated by matching actual financial statement variables to their model analogues. To assess the quality of the approach, the model-implied intrinsic values are subsequently compared to firms' actual market values. Second, the behavior of the financial statement data used in the estimation is compared to the predictions in Corollaries 1 through 6. These tests are not meant to exhaust the potential applications of the model but can serve as a preliminary validation.

The estimation of the model parameters is implemented at the level of the individual firm, based on the available history of each firm's annual revenues, assets and earnings before interest and tax. The emphasis in the following setup is on expositional parsimony and clarity; more sophisticated approaches undoubtedly exist. Investment values and costs are assumed to take the stochastic forms in (24) and (25), respectively, possibly with some additional transient error introduced by the accounting process. By assumption, the state variable $\beta_0(t)$ in (24) and (25) follows a non-stationary process and does not have bounded variance.⁵⁰ To address the resulting consistency problem, all variables are estimated in the form of contemporaneous or between-period ratios, and errors are converted into additive form by taking logarithms. With t defined as measuring time in years, the ratio of current-to-prior-period revenue is thus predicted by

$$\ln\left(\frac{REVT(t)}{REVT(t-1)}\right) = \ln\left(\int_{t-1}^t w(u) du\right) - \ln\left(\int_{t-2}^{t-1} w(u) du\right) \quad (26)$$

⁴⁹ As a straightforward elaboration, one could assign a separate state variable to each investment, potentially correlated across investment vintages. For the objective of this paper, the single state variable version suffices.

⁵⁰ The variance of $\beta_0(t+h)$, conditional on $\beta_0(t)$, is

$$\text{Var}(\beta_0(t+h)|\beta_0(t)) = \beta_0^2(t)(e^{\sigma^2 h} - 1)$$

which increases monotonically in the time distance h .

where $REVT(t)$ is the firm's actual sales revenue reported for the fiscal year ended at time t .⁵¹

Similarly, the year-over-year investment expenditure ratio is given by

$$\ln\left(\frac{INVEST(t)}{INVEST(t-1)}\right) = \ln\left(\int_{t-1}^t k(u) du\right) - \ln\left(\int_{t-2}^{t-1} k(u) du\right) \quad (27)$$

where investment expenditure is calculated as

$$INVEST = REVT - EBIT + \Delta AT$$

and $EBIT$ denotes earnings before interest and tax, and ΔAT denotes the change in the book value of assets between beginning and end of the year.⁵² To improve the efficiency of the estimates, three contemporaneous cross-item ratios are added, including the revenue-to-assets ratio

$$\ln\left(\frac{REVT(t)}{AT(t)}\right) = \ln\left(\int_{t-1}^t w(u) du\right) - \ln(b(t)) \quad (28)$$

where AT denotes the balance of total assets at the end of the fiscal period; the revenue-to-investment ratio

$$\ln\left(\frac{REVT(t)}{INVEST(t)}\right) = \ln\left(\int_{t-1}^t w(u) du\right) - \ln\left(\int_{t-1}^t k(u) du\right) \quad (29)$$

which has incremental value over (26) and (27) because w and k contain different leading coefficients; and the revenue-to-accrual-expense ratio

$$\ln\left(\frac{REVT(t)}{EXP(t)}\right) = \ln\left(\int_{t-1}^t w(u) du\right) - \ln\left(b(t-1) - b(t) + \int_{t-1}^t k(u) du\right) \quad (30)$$

⁵¹ A brief remark is in order on whether the expected residual in (26) is indeed zero. For an indefinite-lived firm, the logarithm of the intertemporal ratio of state variables would have an expected value of

$$E\left(\ln\left(\frac{\beta_0(t+h)}{\beta_0(t)}\right)\right) = \frac{-\sigma^2 h}{2} < 0$$

for any $h > 0$. At the same time, companies that fail tend to do so following a streak of negative growth, which implies a countervailing upward bias in β_0 , conditional on having survived the current period. To understand why one should expect $E(\ln(\beta_0(t+h))) = \ln(\beta_0(t))$ as a net effect, let $\underline{\beta}(t)$ denote the (possibly time-dependent) threshold state below which the firm shuts down and let $T \equiv \inf(t: \beta_0(t) \leq \underline{\beta}(t))$ denote the termination date. The monotone likelihood ratio of the normal distribution implies that $E(\beta_0(t+h)|t+h < T)$ is increasing and concave in $\beta_0(t)$, so that, for any given t and h , there exists a unique level $\beta_0^*(t)$ at which

$$E(\ln(\beta_0(t+h)) | \beta_0(t), t+h < T) = \ln(\beta_0(t))$$

and thus, in expectation, $\ln(\beta_0)$ increases (decreases) toward $\ln(\beta_0^*)$ when $\beta_0 < \beta_0^*$ ($\beta_0 > \beta_0^*$).

⁵² The accrual-basis revenue variable $REVT$ is used here as an approximation of the model's cash revenue variable w . Adjusting $REVT$ by changes in receivables and accrued and deferred revenues does not materially alter the results. The computation of the investment expenditure is likewise only a coarse approximation. Alternative approaches, such as the use of cash flow statement data, are possible but require the availability of additional data items and may limit the sample size.

where $EXP = REVT - EBIT$. The time index in all equations is counting years since the company's founding date.⁵³ The estimation is implemented as a non-linear least-squares minimization of the squared residual differences in (26) through (30), summed over the firm's available history of annual financial statements.⁵⁴ As no analytical solution exists for the first-order conditions of the criterion function, the optimization is carried out numerically.⁵⁵

Summary statistics of the estimated parameter values, based on a total sample of 10,805 firms with an average (median) financial statement history of 18.6 (14) years, are shown in Table 1.⁵⁶ The distribution of the estimated discount rate r is shown by histogram in Figure 5 and has a mean (median) value of 9.9% (9.6%), approximately equal to the weighted-average stock and bond returns over the sample period.⁵⁷ The long-run (asymptotic) growth rate, given by r minus $\beta_2\theta$, has a mean (median) value of 0.014 (0.054), less than nominal long-run macroeconomic growth and consistent with the logic that individual companies cannot outgrow the overall economy indefinitely. The difference between α and β_1 , which measures economic profitability, is less than 10^{-5} for close to half of the sample, suggesting that most companies operate in a fairly competitive environment. Consistent with the hypothesis that high economic profitability should

⁵³ The founding date is standardized to 10 years prior to the company's first appearance in Compustat. Using companies' actual founding dates instead (where available) or estimating the founding date as an additional parameter does not materially affect the results. Note also that the founding date approaches collinearity with β_1 as t increases.

⁵⁴ The first year of each company's data series is omitted from calculations, to avoid the well-known idiosyncrasies surrounding initial public offerings. Further, as is common in non-linear optimization, the distribution of the residuals in (26) through (30) cannot be represented in terms of elementary functions, but the logarithm of integrated geometric Brownian motion can in many cases be closely approximated by a normal distribution (see, for example, Dufresne 2004).

⁵⁵ Solutions are computed via simulated annealing and the Nelder-Mead algorithm. Since the criterion function may have multiple local extrema for any given firm, the computation is repeated with 5 different starting values to reduce the probability of missing the global minimum. When different solutions are returned, the one yielding the lowest squared error value is retained. The computation is carried out subject to the constraints $\alpha \geq \beta_1$ and $\theta, \beta_2 > 0$ implied by the theory, and to the technical constraints $\beta_1 \leq 2$, $\lim_{t \rightarrow \infty} m/w \in (0.5, 5)$ and $\lim_{t \rightarrow \infty} m/b < 10$, which are imposed to prevent the algorithm from diverging toward extreme values. To gauge the restrictiveness the latter two asymptotic constraints, note that less than 10% of firms with more than 40 years of data in Compustat have actual market-to-book or market-to-sales ratios outside the imposed boundaries in their later years.

⁵⁶ The sample consists of firms in Compustat with at least 5 years of revenue and asset data between years 1950 and 2017 and at least one year with an observable stock price. Excluded are (i) entities in financial services (SIC codes 6000 through 6999); (ii) entities whose time series terminate for reasons generally associated with financial distress (Compustat deletion codes 2, 3, 4, 7 and 10); and (iii) firms whose data series show, on average, a sales profit margin below -100%, research and development costs comprising more than 20% of total expenses, or cash and short-term investments comprising more than 40% of total assets. Firms in condition (ii) tend to show a streak of negative growth leading up to the termination date and thus generally violate the mean-zero error condition in β_0 required for unbiased estimates. Firms in condition (iii) generally require an extension of the accrual model to accommodate the immediate expensing of research and development costs and the presence of non-productive financial assets.

⁵⁷ A reasonable fit for the histogram in Figure 5 appears to be the Laplace distribution, which has also been proposed as a good description of the empirical distribution of companies' growth rates (see, for example, Bottazzi and Secchi 2006). Refer to the analysis around Corollary 1 for a discussion of the relationship between r and growth.

be unsustainable over the long run, firms with a profitability value of $\alpha - \beta_1 > 10^{-3}$ have a mean (median) asymptotic growth rate of 0.0009 (0.0413), with a standard error of 0.0018, whereas firms with $\alpha - \beta_1 < 10^{-3}$ show a significantly higher mean (median) long-term growth of 0.0521 (0.0632), with a standard error of 0.0023.⁵⁸

If the model is a good description of a firm's actual value flows and if financial markets are efficient, the intrinsic firm value m implied by these estimation results should equal the firm's actual contemporaneous market values, i.e.,

$$m(t) = \frac{EQ(t)}{1 - tax(t)} + LT(t) \quad (31)$$

where the equity value EQ is calculated as

$$EQ = CSHO \cdot prcc_f + PSTK + MIB$$

from the number of shares outstanding, $CSHO$, the closing share price at the end of the fiscal year, $prcc_f$, preferred stock, $PSTK$, and minority interest, MIB , and LT denotes the total amount of liabilities as reported on the balance sheet.⁵⁹ Income tax effects are not accounted for in the estimation, and so the equity component EQ is inflated to its pre-tax value by the tax rate in effect at the end of the fiscal year.⁶⁰ The state variable $\beta_0(t)$ in $m(t)$ is, by design, not identified by the optimization procedure and is therefore computed separately from the model-to-actual ratios of revenues and costs in the given period.⁶¹ Figure 6 shows a scatter plot of actual against model-implied firm values in logarithmic units. The mean (median) distance between the two values has a signed value of -0.003 (-0.060) and an absolute value of 0.658 (0.565). The standard deviation of 0.842 falls within the range of values obtained under various multiples-based approaches in prior research (see, for example, Liu, Nissim and Thomas 2002).⁶² The product-moment (rank) correlation between actual and model-implied values is 0.922 (0.918). As a refer-

⁵⁸ Since this tradeoff can obtain mechanically when the constraints imposed on m/w and m/b in the estimation bind, only firms with non-binding constraints are included in this calculation.

⁵⁹ For lack of readily available market data, liabilities, minority interest and preferred stock values are assumed to equal their balance sheet amounts.

⁶⁰ The tax rates are linearly interpolated between 50% in 1950 and 28% in 2017. This formula is an approximation of the time series of the aggregate annual ratios of tax expense to pre-tax income across all firms.

⁶¹ Using revenue, one thus obtains $\beta_0(t) = REVT(t)/w(t)$, where the β_0 in w is normalized to unity. The calculations based on investment costs and accrual-basis expenses are analogous. The β_0 used in the left-hand side of (31) is the average of these three values. This approach can be viewed as multiples-based valuation: the estimated parameter values of the model imply expected value multiples (of revenues, investment, etc.), which can then be applied to the company's actual revenue, investment, etc. amounts in the period in question.

⁶² Comparisons to prior research should be prefaced with the general caveat that prior studies typically (i) exclude small firms and/or firms with losses; and (ii) use market value data from peer firms or analyst forecasts as inputs.

ence point, the finite-horizon-forecast-plus-constant-long-term-growth model by Frankel and Lee (1998) yields (rank) correlations between 0.7 and 0.8.

As a second test, one can examine whether the input data used in the estimation exhibit the behavior outlined in Corollaries 1 through 6. To this end, Table 2 gives a summary of the period-to-period changes in seven observable financial ratios discussed in section 3.2. All statistics are computed at the firm level and then aggregated across the same sample used in the estimation above. As predicted in Corollary 1, the average firm's revenue growth rate, calculated as shown on the left-hand side of (26), decreases by -0.0224 from one period to the next, with a standard error of 0.0011 . The ratio of revenue to investment, given by the left-hand side of (29) and predicted by Corollary 2 to increase over time, shows a positive change between periods of 0.0151 on average, with a standard error of 0.0010 . Corollaries 4 and 5 claim that the return on assets is a monotonically increasing function of firm age while the return on sales need not be. Indeed, the return on sales, calculated as shown on the left-hand side of (30), shows an insignificant average period-to-period change of -0.0002 , with a standard error of 0.0006 , whereas the return on assets increases by 0.0016 , albeit with a higher standard error of 0.0011 that does not imply an ROA increase greater than zero at conventional significance levels. Consistent with Corollary 3, the average firm's market-to-sales ratio, calculated as the logarithm of the ratio between the right-hand side of (31) and $REVT$, shows a negative change of -0.0251 from one year to the next, with a standard error of 0.0011 . The market-to-book ratio is likewise decreasing, but at a smaller rate of -0.0179 . (Recall that, by the prediction in Corollary 6, the market-to-book ratio is not generally monotonic.) Finally, to attempt some validation of the model's basic premise that firms' investment life cycles contract over time, one can examine the ratio of firms' net to gross property, plant and equipment, given by the Compustat variables $PPENT$ and $PPEGT$. If depreciation is, on average, matched to the revenues associated with the asset and, as shown in (10), investments' revenue cycles condense as the firm matures, the total net carrying amount of fixed assets should decrease relative to the acquisition cost, and the data indeed indicate an average year-to-year decline by -0.0091 , with a standard error of 0.0002 .

5. Conclusion

A persistent challenge in accounting-based valuation lies in finding a well-motivated description of the future evolution of a firm's investments and profits. Historical financial accounting data

deliver information about inputs and outputs of a firm's value creation process but provide no recipe to infer their intertemporal linkage, which is critical in mapping financial performance into the future. Extant solutions to the problem tend to rely on ad-hoc assumptions, such as perpetual linear profit growth, and external information sources, such as analyst forecasts of future earnings. This paper describes an alternative approach, positing that firms progress through predictable phases of development over time. This life cycle model of the firm yields a description of investing activities and of their associated revenues as a function of the firm's age. While compatible with a typical neoclassical capital accumulation framework, the model does not assume a specific production technology but rather describes a firm's maturing process, over the course of which new investments begin at increasingly advanced stages as the firm moves along its learning curve and competition intensifies.

Estimation of the model requires no analyst forecasts, peer firm data or an exogenously specified cost of capital and can rely solely on firms' own financial statements. The life cycle model thus remains functional even for sparse information sets, such as data at the segment level. The model further yields an implied cost of capital without requiring firms to have an observable market value. The life cycle structure anticipates expenditures to exceed revenues during a firm's startup phase and can therefore rationalize the performance of firms reporting negative cash flows, eliminating the need to remove loss firms from test samples or to make ad-hoc assumptions how these firms become profitable in the future. Empirical tests suggest that firm value estimates align well with observed market values, and that the model's directional predictions about the time-series trends of various financial ratios fit the behavior of actual financial data.

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Appendix

Proof of Proposition 1. Assumption 1 implies that

$$\theta e^{ru} v_h(t - u, \theta u) - e^{ru} v_t(t - u, \theta u) + r e^{ru} v(t - u, \theta u) = 0$$

for all u and t , which reduces to

$$\theta v_h - v_t + r v = 0 \tag{A1}$$

Equation (A1) is a homogeneous first-order, linear partial differential equation with constant coefficients, which has the well-known canonical form

$$v_x(x, y) + \frac{r}{1 + \theta^2} v(x, y) = 0$$

where $x = \theta h - t$ and $y = \theta t + h$. After simplifications, the general solution

$$v(t, h) = e^{rt} f(\theta t + h)$$

obtains, where f is any univariate function such that, for any t , $v_h(t, h) \rightarrow 0$ as $h \rightarrow \infty$. That $f'(\theta t + h) < 0$ at all t and h follows by definition, since v is the present value of all remaining revenues and can therefore only decrease over time.

Proof of Proposition 2. Given Assumption 3, the investment expenditure function can be written as a hypothetical supplier's revenue function

$$k(t) = - \int_{-\infty}^t \frac{\alpha_0 \alpha_3 e^{r(t-u)}}{(1 + e^{\alpha_1 + (\alpha_2 - \alpha_3)t + \alpha_3 u})(1 + e^{-\alpha_1 + (\alpha_2 - \alpha_3)t - \alpha_3 u})} du = \frac{\alpha_0 e^{rt}}{1 + e^{\alpha_1 + \alpha_2 t}}$$

where $\alpha_3 > 0$ and r is the same value as in the firm's value function v .⁶³ The latter claim must hold because, if the hypothetical supplier were to apply a different discount rate $r^S \neq r$, investors would face an arbitrage opportunity, which would be incompatible with the assumption of efficient capital markets. In particular, an investor who acquires both the firm and its supplier would receive a net value flow of

$$w - k + w^S - k^S = w - k^S$$

at each future date, where the firm's investment spending and the supplier's revenue w^S cancel by construction. Hence, it must be that investors value k and w^S identically, which requires that $r^S = r$. Assumption 3 further implies that

⁶³ The equivalent of $\beta_2 \theta$ is collapsed to a single parameter α_2 for notational efficiency.

$$\lim_{t \rightarrow \infty} \frac{v(t, 0)}{k(t)} = \frac{\beta_0}{\alpha_0} \lim_{t \rightarrow \infty} \frac{1 + e^{\alpha_1 + \alpha_2 t}}{1 + e^{\beta_1 + \beta_2 \theta t}} < \infty$$

At the same time, $v(t, 0) \geq k(t)$ is required for all t since the firm would otherwise not invest, and hence must attain a finite but strictly positive limit value, which implies that

$$\alpha_2 = \beta_2 \theta$$

Finally, a zero net present value at time $t = 0$ requires that

$$k(0) = v(t, 0) \Leftrightarrow \frac{\alpha_0}{1 + e^{\alpha_1}} = \frac{\beta_0}{1 + e^{\beta_1}} \Leftrightarrow \alpha_0 = \beta_0 \frac{1 + e^{\alpha_1}}{1 + e^{\beta_1}}$$

which yields the claimed result, with $\alpha = \alpha_1$.

Proof of Corollary 1. The revenue growth rate is zero if

$$\frac{1}{1 + e^{\beta_1 + \beta_2 \theta t}} \left(r - \frac{\beta_2 \theta}{1 + e^{-\beta_1 - \beta_2 \theta t}} \right) = \frac{1}{1 + e^{\beta_1 + \beta_2 t}} \left(r - \frac{\beta_2}{1 + e^{-\beta_1 - \beta_2 t}} \right) \quad (\text{A2})$$

Consider first the case $\theta < 1$. Then the equality can only hold if the parenthetical terms on both sides of (A2) are negative. Rearranging yields

$$\frac{\theta(2 + e^{\beta_1 + \beta_2 t} + e^{-\beta_1 - \beta_2 t})}{2 + e^{\beta_1 + \beta_2 \theta t} + e^{-\beta_1 - \beta_2 \theta t}} = \frac{\theta\beta_2 - \theta(1 + e^{-\beta_1 - \beta_2 t})r}{\theta\beta_2 - (1 + e^{-\beta_1 - \beta_2 \theta t})r} \quad (\text{A3})$$

where all numerator and denominator terms are positive. If, at any t at which (A3) holds, the change in t in the right-hand side exceeds the change in left-hand side, the revenue growth rate is decreasing around this t . The right-hand side of (A3) is always above unity and is decreasing in t . The left-hand side of (A3) is monotonically increasing in t at values above unity, since then

$$e^{\beta_1 + \beta_2 t} - e^{-\beta_1 - \beta_2 t} > (e^{\beta_1 + \beta_2 \theta t} - e^{-\beta_1 - \beta_2 \theta t}) \frac{\theta(2 + e^{\beta_1 + \beta_2 t} + e^{-\beta_1 - \beta_2 t})}{2 + e^{\beta_1 + \beta_2 \theta t} + e^{-\beta_1 - \beta_2 \theta t}}$$

and thus there can exist at most one crossing point. To determine existence, it suffices to examine the limit value of the growth rate, which is negative for any $r < \beta_2 \theta$. Since r only affects the growth rate as an additive constant, the location of the root can be shifted to any value of t by changing r , and hence the growth rate must be decreasing at all t . For the case $\theta > 1$, one can replace all terms $\beta_2 \theta$ with β_2 and vice versa and repeat the argument.

Proof of Corollary 2. A net cash flow of zero obtains when

$$w(t) = k(t) \Leftrightarrow \frac{1 + e^{\alpha + \beta_2 \theta t}}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{1 + e^{\alpha + \beta_2 t}}{1 + e^{\beta_1 + \beta_2 t}} = \frac{(1 - \theta)(1 + e^\alpha)}{1 + e^{\beta_1}} \quad (\text{A4})$$

Consider first the case $\theta < 1$. Existence of a value $t = T$ such that (A4) holds follows from the observation that the left-hand side of the second equality in (A4) is zero at $t = 0$ but diverges to ∞ as $t \rightarrow \infty$. Uniqueness follows from the observation that

$$\frac{1 + e^{\alpha + \beta_2 \theta t}}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{1 + e^{\alpha + \beta_2 t}}{1 + e^{\beta_1 + \beta_2 t}} = \frac{1 + e^{\alpha + \beta_2 \theta t}}{1 + e^{\beta_1 + \beta_2 \theta t}} \left(1 - \frac{1 + e^{\beta_1 + \beta_2 t}}{1 + e^{\beta_1 + \beta_2 \theta t}} \right)$$

is monotonically increasing in t , in view of $\alpha \geq \beta_1$ and $\beta_2 \theta < \beta_2$. The argument is symmetric for the case $\theta > 1$. That T is decreasing in α follows from inspection of (A4), as $(1 + e^{\alpha + \beta_2 \theta t})$ increases in α faster than $(1 + e^\alpha)$, so that maintaining (A4) in response to an increase in α requires a lower t . To determine the effect of θ on T , rearrange (A4) to

$$\frac{1}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{1}{1 + e^{\beta_1 + \beta_2 t}} = \frac{(1 - \theta)(1 + e^\alpha)}{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta t})} \quad (\text{A5})$$

and observe that first fraction on the left-hand side decreases in θ proportionately slower than the right-hand side, in view of $\alpha \geq \beta_1$. Hence, there can exist at most one θ below which the left-hand side is decreasing by more than the right-hand side. But since, at $\theta = 0$, the increase in the two sides of (A5) is

$$-\frac{1}{1 + e^{-\beta_1}} > -1 - \frac{1}{1 + e^{-\alpha}}$$

it must be that the left-hand side of (A5) is decreasing less than the right-hand side at all θ .

Hence, maintaining the equality in (A5) in response to an increase in θ requires a decrease in T .

Regarding the second claim in Corollary 2, the ratio of w to k is

$$\frac{w(t)}{k(t)} = \left(\frac{1}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{1}{1 + e^{\beta_1 + \beta_2 t}} \right) \frac{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta t})}{(1 - \theta)(1 + e^\alpha)}$$

The value at $t = 0$ obtains by direct evaluation. The limit values

$$\lim_{t \rightarrow \infty} \frac{w(t)}{k(t)} = \frac{1 + e^{-\beta_1}}{(1 - \theta)(1 + e^{-\alpha})}$$

for $\theta < 1$ and ∞ for $\theta \geq 1$ obtain by direct computation. For $\theta < 1$, the ratio is increasing in t if

$$\theta \frac{e^{\alpha + \beta_2 \theta t} - e^{\beta_1 + \beta_2 \theta t}}{(1 + e^{\beta_1 + \beta_2 \theta t})^2} > \frac{\theta(1 + e^{\beta_1 + \beta_2 t})e^{\alpha + \beta_2 \theta t} - e^{\beta_1 + \beta_2 t}(1 + e^{\alpha + \beta_2 \theta t})}{(1 + e^{\beta_1 + \beta_2 t})^2}$$

The left-hand side is always positive since $\alpha \geq \beta_1$, so the inequality holds whenever the right-hand side is negative. If the right-hand side is positive, its magnitude must be smaller than that of the left-hand side because of $\theta < 1$ and $\alpha \geq \beta_1$. For $\theta > 1$, the inequality is reversed and the argument is symmetric.

Proof of Corollary 3. The market-to-sales ratio changes in t by

$$\frac{d}{dt} \left(\frac{m(t)}{w(t)} \right) = \frac{m'(t)}{w(t)} - \frac{w'(t)m(t)}{w^2(t)}$$

which is positive if

$$\frac{m'(t)}{m(t)w(t)} = \frac{k(t) - w(t)}{m(t)w(t)} > \frac{w'(t)}{w^2(t)}$$

As a function of α , the left-hand side is maximized when

$$\frac{k'(t)}{k(t)} = r - \frac{\beta_2 \theta}{1 + e^{-\alpha - \beta_2 \theta t}}$$

attains its minimum, i.e., when $\alpha = \beta_1$. Hence, if the market-to-sales ratio is monotonically decreasing in t for $\alpha = \beta_1$, it must be decreasing for all α . Given $\alpha = \beta_1$, one obtains

$$\frac{m(t)}{w(t)} \propto \frac{\ln(1 + e^{-\beta_1 - \beta_2 \theta t}) - \ln(1 + e^{-\beta_1 - \beta_2 t})}{\frac{1}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{1}{1 + e^{\beta_1 + \beta_2 t}}}$$

and

$$\frac{m'(t)}{w'(t)} \propto \frac{\frac{1}{1 + e^{\beta_1 + \beta_2 t}} - \frac{\theta}{1 + e^{\beta_1 + \beta_2 \theta t}}}{\frac{1}{(1 + e^{\beta_1 + \beta_2 t})(1 + e^{-\beta_1 - \beta_2 t})} - \frac{\theta}{(1 + e^{\beta_1 + \beta_2 \theta t})(1 + e^{-\beta_1 - \beta_2 \theta t})}} \quad (\text{A6})$$

which both evaluate to $1 + e^{-\beta_1}$ at $t = 0$ and to 1 as $t \rightarrow \infty$. Consider the case $\theta < 1$. The argument for the case $\theta > 1$ is symmetric. If (A6) is negative, it must be that $m' < 0$ while $w' > 0$ and hence the market-to-sales ratio must be decreasing in t . If (A6) is positive and equal to some value $z > 0$, one obtains

$$\frac{\theta}{1 + e^{\beta_1 + \beta_2 \theta t}} \left(1 - \frac{z}{1 + e^{-\beta_1 - \beta_2 \theta t}} \right) = \frac{1}{1 + e^{\beta_1 + \beta_2 t}} \left(1 - \frac{z}{1 + e^{-\beta_1 - \beta_2 t}} \right) \quad (\text{A7})$$

If (A6) is positive in numerator and denominator, both parenthetical terms in (A7) must be positive. Then the left-hand side decreases in t less than the right-hand side, which implies that (A6) is decreasing in t . If (A6) is negative in numerator and denominator, one can write

$$\frac{2 + e^{\beta_1 + \beta_2 \theta t} + e^{-\beta_1 - \beta_2 \theta t}}{2 + e^{\beta_1 + \beta_2 t} + e^{-\beta_1 - \beta_2 t}} = \theta \frac{z - 1 - e^{-\beta_1 - \beta_2 \theta t}}{z - 1 - e^{-\beta_1 - \beta_2 t}}$$

where numerator and denominator on the right-hand side must both be positive, which implies that the right-hand side is less than θ . Then the left-hand side fraction must be decreasing in t as

$$e^{\beta_1 + \beta_2 \theta t} - e^{-\beta_1 - \beta_2 \theta t} \leq e^{\beta_1 + \beta_2 t} - e^{-\beta_1 - \beta_2 t}$$

for $\theta < 1$, while the right-hand side fraction is increasing in t , and so (A6) is again decreasing in t . Then the market-to-sales ratio can at most be increasing at low t , but since, in view of the limit properties noted above, the change in the ratio is zero at $t = 0$, the market-to-sales ratio must be decreasing monotonically in t . The limit value of the ratio is

$$\lim_{t \rightarrow \infty} \frac{m(t)}{w(t)} = \lim_{t \rightarrow \infty} \frac{m'(t)}{w'(t)} = \frac{1}{\beta_2} \lim_{t \rightarrow \infty} \frac{\frac{1}{1+e^{\beta_1+\beta_2 t}} - \frac{1}{1+e^{\beta_1+\beta_2 \theta t}} + \frac{(1-\theta)(1+e^\alpha)}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2 \theta t})}}{\frac{1}{(1+e^{\beta_1+\beta_2 t})(1+e^{-\beta_1-\beta_2 t})} - \frac{\theta}{(1+e^{\beta_1+\beta_2 \theta t})(1+e^{-\beta_1-\beta_2 \theta t})}}$$

by L'Hôpital's rule, and thus

$$\lim_{t \rightarrow \infty} \frac{m(t)}{w(t)} = \frac{1}{\beta_2 \theta} \left(1 - \frac{(1-\theta)(1+e^{-\alpha})}{1+e^{-\beta_1}} \right)$$

when $\theta < 1$ and

$$\lim_{t \rightarrow \infty} \frac{m(t)}{w(t)} = \frac{1}{\beta_2}$$

when $\theta \geq 1$.

Proof of Corollary 4. The common coefficient β_0 cancels between p and b and will be omitted from the notation in this proof. After replacing v and k in (16) and (17) with their parametric solutions, one can write earnings and book value as

$$\begin{aligned} p(t) &= \int_0^t \frac{\beta_2 \left(e^{rt} - e^{ru} \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2 \theta u})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2 \theta u})} \right)}{(1+e^{\beta_1+\beta_2(t-(1-\theta)u)})(1+e^{-\beta_1-\beta_2(t-(1-\theta)u)})} du \\ &= \int_0^1 \frac{\beta_2 t \left(e^{rt} - e^{rtu} \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2 \theta tu})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2 \theta tu})} \right)}{(1+e^{\beta_1+\beta_2 t(1-(1-\theta)u)})(1+e^{-\beta_1-\beta_2 t(1-(1-\theta)u)})} du \end{aligned}$$

and

$$b(t) = \int_0^t \frac{e^{ru}(1+e^\alpha)(1+e^{\beta_1+\beta_2 \theta u})}{(1+e^{\beta_1})(1+e^{\beta_1+\beta_2(t-(1-\theta)u)})(1+e^{\alpha+\beta_2 \theta u})} du = \int_0^1 \frac{t e^{rtu} \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2 \theta tu})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2 \theta tu})}}{1+e^{\beta_1+\beta_2 t(1-(1-\theta)u)}} du$$

where the final equalities obtain after substitution of variables. The common factor t cancels and can be ignored. Consider first the case $\theta \leq 1$ and assume $r = 0$ for the moment. That the ratio of p to b is monotonically increasing in t then follows from four observations. First, the numerator of the integrand in p is increasing in t and u while the numerator in b is decreasing. Second, the relative weighting by the common denominator term $(1+e^{\beta_1+\beta_2 t(1-(1-\theta)u)})$ in p and b shifts

monotonically from low to high u as t increases, which increases the ratio of p to b because the ratio of the numerators of the integrands is increasing in u . Third, the denominator term $(1 + e^{-\beta_1 - \beta_2 t(1-(1-\theta)u)})$ in p is increasing in t , and increasing t also shifts weight monotonically toward higher u . Fourth, for $r > 0$, higher t in e^{rt} and e^{rtu} increases the ratio of the integrands at every u . For $\theta > 1$, begin with $r = 0$ and integrate by parts to obtain

$$p_t(t) = \frac{\beta_2 \left(1 - \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2\theta t})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2\theta t})} \right)}{(\theta-1)(1+e^{\beta_1+\beta_2\theta t})(1+e^{-\beta_1-\beta_2\theta t})} + \int_0^t \frac{\beta_2 \frac{d}{du} \left(1 - \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2\theta u})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2\theta u})} \right)}{(\theta-1)(1+e^{\beta_1+\beta_2(t-(1-\theta)u)})(1+e^{-\beta_1-\beta_2(t-(1-\theta)u)})} du$$

and

$$b_t(t) = \frac{\frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2\theta t})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2\theta t})}}{(\theta-1)(1+e^{\beta_1+\beta_2\theta t})} + \int_0^t \frac{\frac{d}{du} \left(\frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2\theta u})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2\theta u})} \right)}{(\theta-1)(1+e^{\beta_1+\beta_2(t-(1-\theta)u)})} du$$

The integral terms in p_t and b_t are positive and negative, respectively. The ratio of the respective leading terms is the maximum integrand ratio of p and b . Hence, the ratio of p to b is increasing in t . The result extends to $r > 0$ by convexity. That the ratio is increasing in r and α follows from inspection. That the ratio value is zero at $t = 0$ follows from direct evaluation of p and b above, after cancellation of the common factor t . To determine the limit value of the ratio as $t \rightarrow \infty$, consider first the case $\theta > 1$. After multiplying p and b by $e^{\beta_1+(\beta_2-r)t}$, one obtains

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{\beta_1+(\beta_2-r)t} p(t) &= \lim_{t \rightarrow \infty} \int_0^t \frac{\beta_2 e^{\beta_1+(\beta_2-r)t} \left(e^{rt} - e^{ru} \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2\theta u})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2\theta u})} \right)}{(1+e^{\beta_1+\beta_2(t-(1-\theta)u)})(1+e^{-\beta_1-\beta_2(t-(1-\theta)u)})} du \\ &= \lim_{t \rightarrow \infty} \int_0^t \beta_2 e^{\beta_2(1-\theta)u} \left(1 - e^{r(u-t)} \frac{(1+e^\alpha)(1+e^{\beta_1+\beta_2\theta u})}{(1+e^{\beta_1})(1+e^{\alpha+\beta_2\theta u})} \right) du = \frac{1}{\theta-1} \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{\beta_1+(\beta_2-r)t} b(t) &\propto \lim_{t \rightarrow \infty} \int_0^t \frac{e^{\beta_1+(\beta_2-r)t+ru} (1+e^{\beta_1+\beta_2\theta u})}{(1+e^{\beta_1+\beta_2(t-(1-\theta)u)})(1+e^{\alpha+\beta_2\theta u})} du \\ &= \lim_{t \rightarrow \infty} \int_0^t \frac{e^{\beta_2(1-\theta)u+r(u-t)} (1+e^{\beta_1+\beta_2\theta u})}{1+e^{\alpha+\beta_2\theta u}} du = \lim_{t \rightarrow \infty} \frac{e^{-rt} - e^{-\beta_2(\theta-1)t}}{\beta_2(\theta-1) - r} = 0 \end{aligned}$$

and thus the return ratio diverges. For $\theta < 1$, multiply both p and b by $e^{\beta_1+(\beta_2\theta-r)t}$ to obtain

$$\begin{aligned}
\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} p(t) &= \lim_{t \rightarrow \infty} \int_0^1 \frac{\beta_2 t e^{\beta_1 + (\beta_2 \theta - r)t} \left(e^{rt} - e^{rtu} \frac{(1+e^\alpha)(1+e^{\beta_1 + \beta_2 \theta tu})}{(1+e^{\beta_1})(1+e^{\alpha + \beta_2 \theta tu})} \right)}{(1 + e^{\beta_1 + \beta_2 t(1-(1-\theta)u)})(1 + e^{-\beta_1 - \beta_2 t(1-(1-\theta)u)})} du \\
&= \lim_{t \rightarrow \infty} \int_0^1 \beta_2 t e^{\beta_2(\theta-1)t(1-u)} \left(1 - e^{rt(u-1)} \frac{(1 + e^\alpha)(1 + e^{\beta_1 + \beta_2 \theta tu})}{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta tu})} \right) du \\
&= \lim_{t \rightarrow \infty} \int_0^1 \beta_2 t e^{\beta_2(\theta-1)t(1-u)} du - \frac{1 + e^{-\alpha}}{1 + e^{-\beta_1}} \lim_{t \rightarrow \infty} \int_0^1 \beta_2 t e^{(\beta_2(\theta-1)-r)t(1-u)} du \\
&= \frac{1}{1 - \theta} - \frac{\beta_2(1 + e^{-\alpha})}{(1 + e^{-\beta_1})(\beta_2(1 - \theta) + r)}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} b(t) &= \lim_{t \rightarrow \infty} \int_0^1 \frac{t e^{\beta_1 + \beta_2 \theta t + rt(u-1)} (1 + e^\alpha)(1 + e^{\beta_1 + \beta_2 \theta tu})}{(1 + e^{\beta_1 + \beta_2 t(1-(1-\theta)u)})(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta tu})} du \\
&= \lim_{t \rightarrow \infty} \int_0^1 t e^{(\beta_2(1-\theta) + r)t(u-1)} \frac{(1 + e^\alpha)(1 + e^{\beta_1 + \beta_2 \theta tu})}{(1 + e^{\beta_1})(1 + e^{\alpha + \beta_2 \theta tu})} du \\
&= \frac{1 + e^{-\alpha}}{1 + e^{-\beta_1}} \lim_{t \rightarrow \infty} \int_0^1 t e^{(\beta_2(1-\theta) + r)t(u-1)} du = \frac{1 + e^{-\alpha}}{(1 + e^{-\beta_1})(\beta_2(1 - \theta) + r)}
\end{aligned}$$

The ratio of these limit values yields the claimed result.

Proof of Corollary 5. The common coefficient β_0 cancels between p and w and will be omitted from the notation in this proof. That the ratio is increasing in r and α follows from inspection.

The existence of non-monotonic ratios, as a function of t , follows from the numerical examples in Figure 3. To establish that the ratio value is zero at $t = 0$, substitute variables to obtain

$$p(t) = \int_0^1 \frac{\beta_2 t \left(e^{rt} - e^{rtu} \frac{(1+e^\alpha)(1+e^{\beta_1 + \beta_2 \theta tu})}{(1+e^{\beta_1})(1+e^{\alpha + \beta_2 \theta tu})} \right)}{(1 + e^{\beta_1 + \beta_2 t(1-(1-\theta)u)})(1 + e^{-\beta_1 - \beta_2 t(1-(1-\theta)u)})} du$$

and

$$w(t) = \int_0^1 \frac{\beta_2 t e^{rt}}{(1 + e^{\beta_1 + \beta_2 t(1-(1-\theta)u)})(1 + e^{-\beta_1 - \beta_2 t(1-(1-\theta)u)})} du$$

After cancellation of the common factor t , the claim follows from direct evaluation at $t = 0$. To obtain the limit value as $t \rightarrow \infty$, observe that

$$\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 - r)t} w(t) = \frac{1}{1 - \theta} \lim_{t \rightarrow \infty} \left(\frac{e^{\beta_1 + \beta_2 t}}{1 + e^{\beta_1 + \beta_2 \theta t}} - \frac{e^{\beta_1 + \beta_2 t}}{1 + e^{\beta_1 + \beta_2 t}} \right) = \frac{1}{\theta - 1}$$

for $\theta > 1$ and, similarly,

$$\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} w(t) = \frac{1}{1 - \theta}$$

for $\theta < 1$. Taking the corresponding limits for p , whose results are give in the proof of Corollary 4, and dividing yields the claimed result.

Proof of Corollary 6. The common coefficient β_0 cancels between m and b and will be omitted from the notation in this proof. The existence of non-monotonic ratios, as a function of t , follows from the numerical examples in Figure 4. The claim that the ratio of m to b increases in r follows from inspection. An increase in α yields

$$m_\alpha(t) \propto \frac{1 + e^\alpha}{1 + e^{\alpha + \beta_2 \theta t}} - e^\alpha \ln(1 + e^{-\alpha - \beta_2 \theta t})$$

which changes in t by

$$m_{\alpha t}(t) \propto -\frac{1 + e^\alpha}{(1 + e^{\alpha + \beta_2 \theta t})(1 + e^{-\alpha - \beta_2 \theta t})} + \frac{e^\alpha}{1 + e^{\alpha + \beta_2 \theta t}} \leq 0$$

since $t \geq 0$. Then m_α is minimized as $t \rightarrow \infty$, which yields

$$\lim_{t \rightarrow \infty} (1 + e^{\alpha + \beta_2 \theta t}) m_\alpha(t) = \frac{1}{\beta_2 \theta (1 + e^{\beta_1})} > 0$$

after application of L'Hôpital's rule. The effect of an increase in α on book value is

$$b_\alpha(t) \propto \int_0^t \frac{e^{ru} (1 + e^{\beta_1 + \beta_2 \theta u})}{(1 + e^{\beta_1 + \beta_2 (t - (1 - \theta)u)}) (1 + e^{\alpha + \beta_2 \theta u})} \left(e^\alpha - \frac{1 + e^\alpha}{1 + e^{-\alpha - \beta_2 \theta u}} \right) du \leq 0$$

and thus the market-to-book ratio increases in α . That, if $\alpha > \beta_1$, the ratio diverges at $t = 0$ follows from $b(0) = 0$ and $m(0) > 0$. For $\alpha = \beta_1$, apply L'Hôpital's rule to obtain

$$\frac{m(0)}{b(0)} = \frac{m_t(0)}{b_t(0)} = 1$$

after direct evaluation of $m_t(0)$ and $b_t(0)$. To determine the limit value of the ratio as $t \rightarrow \infty$, consider first the case $\theta \geq 1$. After multiplying by $e^{\beta_1 + (\beta_2 - r)t}$ and applying L'Hôpital's rule, one obtains

$$\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 - r)t} m(t) = -\frac{1}{\beta_2} \lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 - r)t} m_t(t) = \frac{1}{\beta_2 (\theta - 1)}$$

and zero for b , as shown in the proof of Proposition 3, and so the market-to-book ratio diverges as $t \rightarrow \infty$.⁶⁴ For $\theta < 1$, multiply by $e^{\beta_1 + (\beta_2 \theta - r)t}$ and apply L'Hôpital's rule to obtain

⁶⁴ As a side note, the case $r < 0$ ($r > 0$ is assumed in the main text) yields a divergent limit value

$$\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} m(t) = -\frac{1}{\beta_2 \theta} \lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} (m_t(t) - r m(t)) = \frac{1}{\beta_2 \theta} \left(\frac{1}{1 - \theta} - \frac{1 + e^{-\alpha}}{1 + e^{-\beta_1}} \right)$$

and

$$\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} b(t) = \frac{1 + e^{-\alpha}}{(1 + e^{-\beta_1})(\beta_2(1 - \theta) + r)}$$

where the limit value of b obtains as in the proof of Corollary 4.⁶⁵ Dividing the two limit values and simplifying yields the claimed result.

Lemma 1. *The ratio of investment expenditures k to gross book value \bar{b} is decreasing monotonically in t . The ratio of gross book value \bar{b} to net book value b is increasing monotonically in t .*

Proof: The common coefficient β_0 cancels in any ratio between k , b and \bar{b} and will be omitted from the notation in this proof. Regarding the ratio \bar{b}/k , observe that $k(t)$ is identical in functional form to $v(t)$ and so, in view of (12),

$$\frac{d}{dt} \left(\frac{k'(t)}{k(t)} \right) < 0$$

for all t , which implies that

$$\frac{k(t)}{\bar{b}(t)} = \left(k(0) + \int_0^t k'(u) du \right) \left(\int_0^t k(u) du \right)^{-1}$$

is decreasing in t . For the ratio b/\bar{b} , apply substitution of variables to its reciprocal to obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} b(t) &\propto \lim_{t \rightarrow \infty} \int_0^t \frac{e^{\beta_2(1-\theta)u + r(u-t)} (1 + e^{\beta_1 + \beta_2 \theta u})}{1 + e^{\alpha + \beta_2 \theta u}} du \geq \lim_{t \rightarrow \infty} \int_0^t e^{\beta_2(1-\theta)u + r(u-t)} du \\ &= \lim_{t \rightarrow \infty} \frac{e^{-rt} - e^{-\beta_2(\theta-1)t}}{\beta_2(\theta-1) - r} = \infty \end{aligned}$$

and thus an asymptotic market-to-book ratio of zero. The special case $r = 0$ then yields

$$\lim_{t \rightarrow \infty} \frac{m(t)}{b(t)} \in \left(1, \frac{1 + e^{-\beta_1}}{1 + e^{-\alpha}} \right)$$

⁶⁵ As a side note, for $r < \beta_2(\theta - 1) < 0$ (the text assumes $r > 0$), book value becomes

$$\lim_{t \rightarrow \infty} e^{\beta_1 + (\beta_2 \theta - r)t} b(t) \geq \lim_{t \rightarrow \infty} \int_0^t \frac{e^{\beta_1 + \beta_2 \theta t + r(u-t)} (1 + e^{\beta_1 + \beta_2 \theta u})}{(1 + e^{\beta_1 + \beta_2(t-(1-\theta)u)}) (1 + e^{\alpha + \beta_2 \theta u})} du \geq \lim_{t \rightarrow \infty} \int_0^t \frac{e^{\beta_1 + \beta_2(t-(1-\theta)u)}}{1 + e^{\beta_1 + \beta_2(t-(1-\theta)u)}} du = \infty$$

and thus the market-to-book ratio diverges.

$$\begin{aligned}\frac{\bar{b}(t)}{b(t)} &= \int_0^t \frac{e^{ru}}{1 + e^{\alpha + \beta_2 \theta u}} du \left(\int_0^t \frac{e^{ru}(1 + e^{\beta_1 + \beta_2 \theta u})}{(1 + e^{\beta_1 + \beta_2(t - (1 - \theta)u)})(1 + e^{\alpha + \beta_2 \theta u})} du \right)^{-1} \\ &= \int_0^1 \frac{e^{rtu}}{1 + e^{\alpha + \beta_2 \theta tu}} du \left(\int_0^1 \frac{e^{rtu}(1 + e^{\beta_1 + \beta_2 \theta tu})}{(1 + e^{\beta_1 + \beta_2 t(1 - (1 - \theta)u)})(1 + e^{\alpha + \beta_2 \theta tu})} du \right)^{-1}\end{aligned}$$

and observe that the terms containing α are identical between \bar{b} and b and are decreasing in u .

Raising t shifts weight toward low u since

$$\frac{\frac{d}{dt} \left(\frac{1}{1 + e^{\alpha + \beta_2 \theta tu}} \right)}{\frac{1}{1 + e^{\alpha + \beta_2 \theta tu}}} = - \frac{\beta_2 \theta u}{1 + e^{-\alpha - \beta_2 \theta tu}}$$

is decreasing in u . The net carrying percentage, given by the ratio of the terms involving β_1 , is monotonically increasing in u , so that higher t increases the ratio of \bar{b} to b through the α -terms.

Further, the net carrying percentage is decreasing in t at all u since

$$\begin{aligned}\frac{d}{dt} \left(\frac{1 + e^{\beta_1 + \beta_2 \theta tu}}{1 + e^{\beta_1 + \beta_2 t(1 - (1 - \theta)u)}} \right) \\ = \frac{\beta_2 \theta u e^{\beta_1 + \beta_2 \theta tu} (1 + e^{\beta_1 + \beta_2 t(\theta u + 1 - u)})}{(1 + e^{\beta_1 + \beta_2 t(\theta u + 1 - u)})^2} \left(1 - \frac{(\theta u + 1 - u)(1 + e^{-\beta_1 - \beta_2 \theta tu})}{\theta u (1 + e^{-\beta_1 - \beta_2 t(\theta u + 1 - u)})} \right) < 0\end{aligned}$$

Lastly, raising t when $r > 0$ shifts the weighting by the term e^{rtu} monotonically toward higher u , where carrying percentages are higher. The effect increases monotonically in r , and since

$$\lim_{r \rightarrow \infty} \frac{\bar{b}(t)}{b(t)} = 1$$

for all t , the change in the ratio with respect to t has an upper bound of zero.

Table 1. Summary of estimated model parameter values, based on a sample of 10,805 firms with an average history of 18.6 fiscal years of data.

	<u>mean</u>	<u>median</u>	<u>std. deviation</u>
θ	0.047	0.010	0.106
r	0.099	0.096	0.117
β_1	-2.620	-0.975	6.566
β_2	2.409	1.786	2.557
α	-2.470	-0.970	6.698

Table 2. Summary statistics of year-to-year changes in seven financial ratios, based on actual reported values from a sample of 10,805 firms. All values are calculated as period-to-period differences in the indicated ratio. In any fiscal year t , revenue growth is computed as $\ln\left(\frac{REVT(t)}{REVT(t-1)}\right)$, the revenue-to-investment ratio as $\ln\left(\frac{REVT(t)}{INVEST(t)}\right)$, the return-on-sales as $\ln\left(\frac{REVT(t)}{REVT(t)-EBIT(t)}\right)$, the return-on-assets as $\ln\left(\frac{REVT(t)}{REVT(t)-EBIT(t)} \cdot \frac{REVT(t)}{AT(t)}\right)$, the market-to-sales ratio as $\ln\left(\frac{MV(t)}{REVT(t)}\right)$, the market-to-book ratio as $\ln\left(\frac{MV(t)}{AT(t)}\right)$, and the net-to-gross fixed assets ratio as $\ln\left(\frac{PPE(t)}{PPEGT(t)}\right)$, where the market value MV is given by the right-hand side of (31).

<u>Period-to-period change in</u>	<u>mean</u>	<u>median</u>	<u>std. error</u>	<u>firm-years</u>
Revenue growth rate	-0.0224	-0.0098	0.0011	178,182
Revenue-to-investment	0.0151	0.0009	0.0010	175,559
Return-on-sales	-0.0002	0.0001	0.0006	188,747
Return-on-assets	0.0016	0.0024	0.0011	188,747
Market-to-sales	-0.0251	-0.0109	0.0011	161,381
Market-to-book	-0.0179	-0.0024	0.0008	161,381
Net-to-gross fixed assets	-0.0091	-0.0122	0.0002	187,052

Figure 1. The revenue flow, $-e^{rh}v_h(t, h)$, of an investment made at time t as a function of the investment age h , and the contemporaneous residual present value of the remaining future revenues, $e^{rh}v(t, h)$. The underlying parametric version of v is given by (6), with $r = 0.1$, $\theta = 0.3$, $\beta_0 = 1$, $\beta_1 = -4$, $\beta_2 = 1.5$, and $t = 0$.

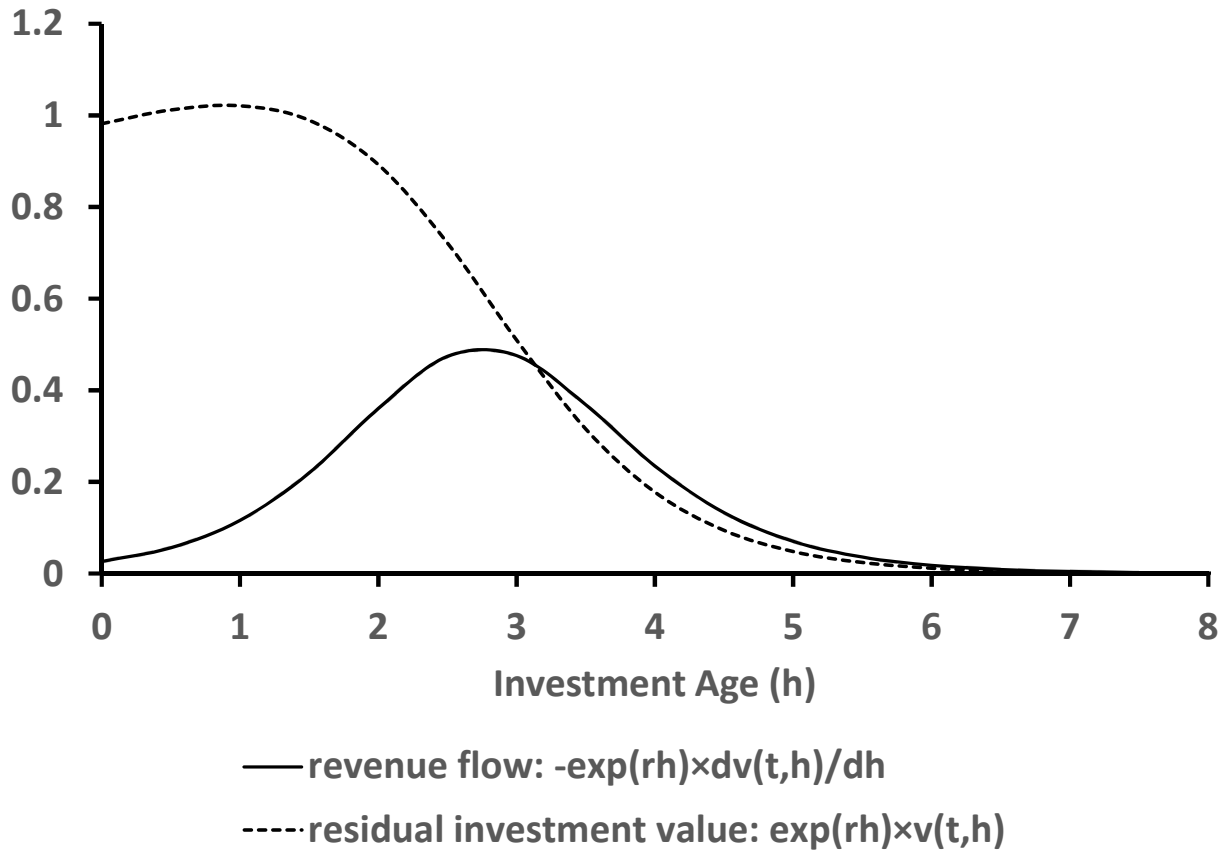


Figure 2. The revenue flow, $-e^{rh}v_h(t, h)$, of an investment made at time t as a function of the investment age h , and the contemporaneous accrual-basis expense, $k(t) \frac{v_h(t, h)}{v(t, 0)}$. The underlying parameter values are $r = 0.1$, $\theta = 0.3$, $\beta_0 = 8$, $\beta_1 = -0.5$, $\beta_2 = 0.5$, $\alpha = 0$ and $t = 0$.

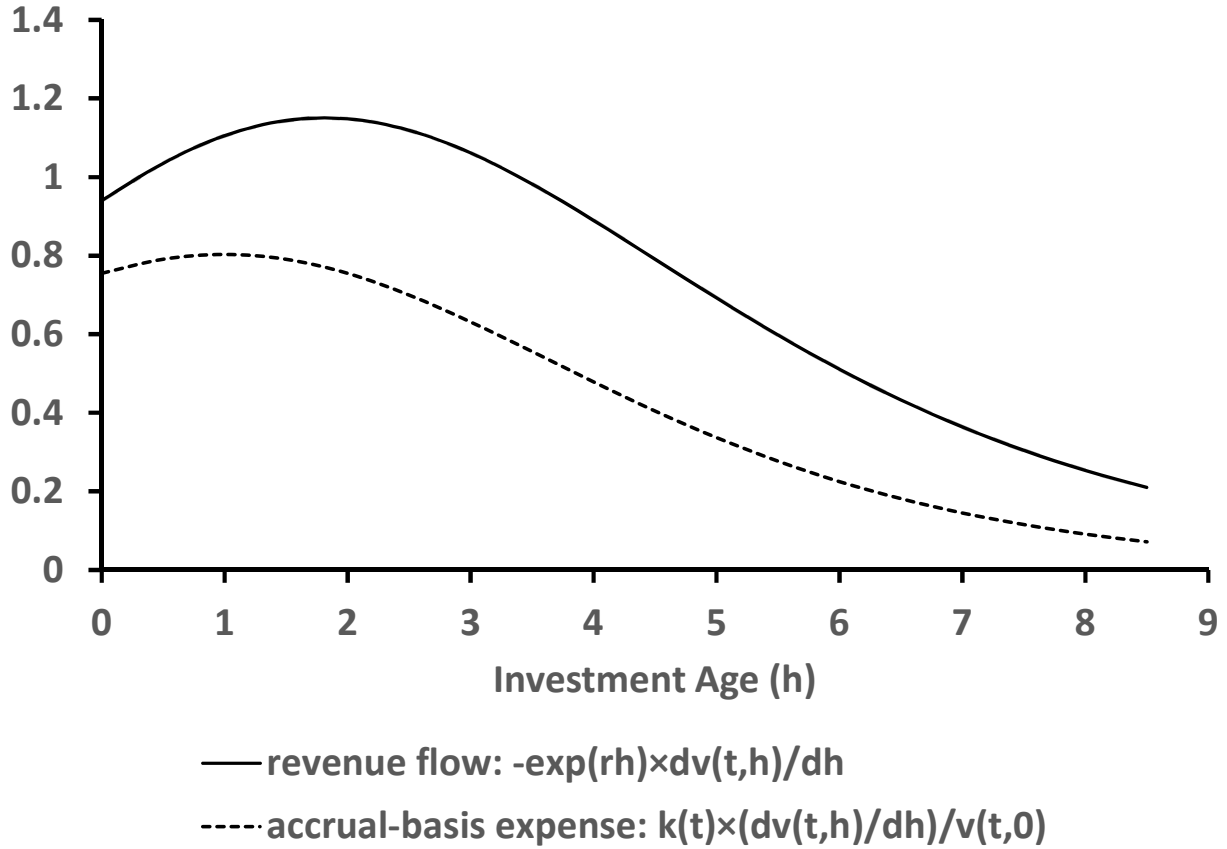


Figure 3. Return on sales, p/w , as a function of firm age t , with parameter values $\theta = 0.2$, $r = 0.05$, $\beta_1 = -5$, $\beta_2 = 0.4$ and $\alpha = -4.8$ in scenario 1; $\theta = 0.5$, $r = 0.2$, $\beta_1 = -3$, $\beta_2 = 0.3$ and $\alpha = -2.9$ in scenario 2; and $\theta = 0.5$, $r = 0.1$, $\beta_1 = -2$, $\beta_2 = 0.3$ and $\alpha = -1.9$ in scenario 3.

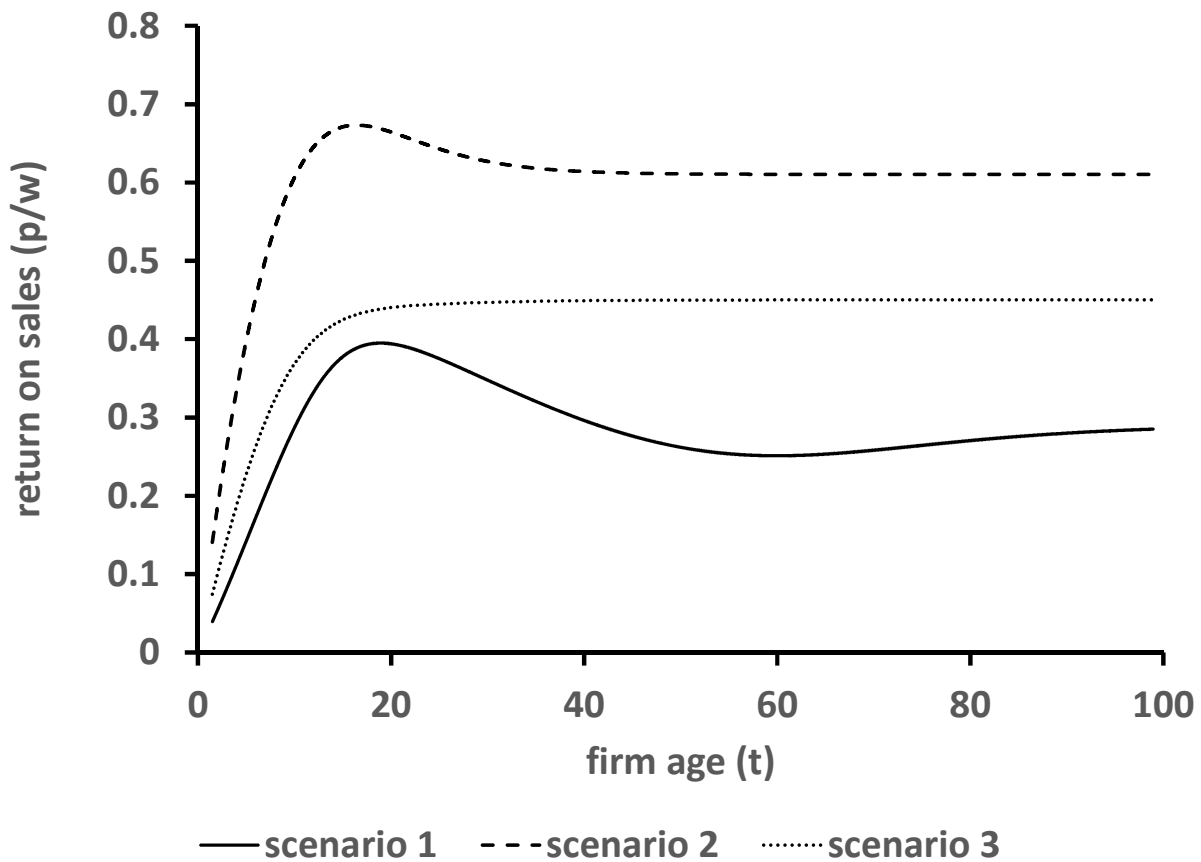


Figure 4. The market-to-book ratio, m/b , as a function of firm age t , with parameter values $\theta = 0.5$, $r = 0.1$, $\beta_1 = -5$, $\beta_2 = 0.2$ and $\alpha = -4.8$ in scenario 1; $\theta = 0.8$, $r = 0.1$, $\beta_1 = -5$, $\beta_2 = 0.3$ and $\alpha = -4.8$ in scenario 2; and $\theta = 0.9$, $r = 0$, $\beta_1 = 2$, $\beta_2 = 0.1$ and $\alpha = 2.5$ in scenario 3.

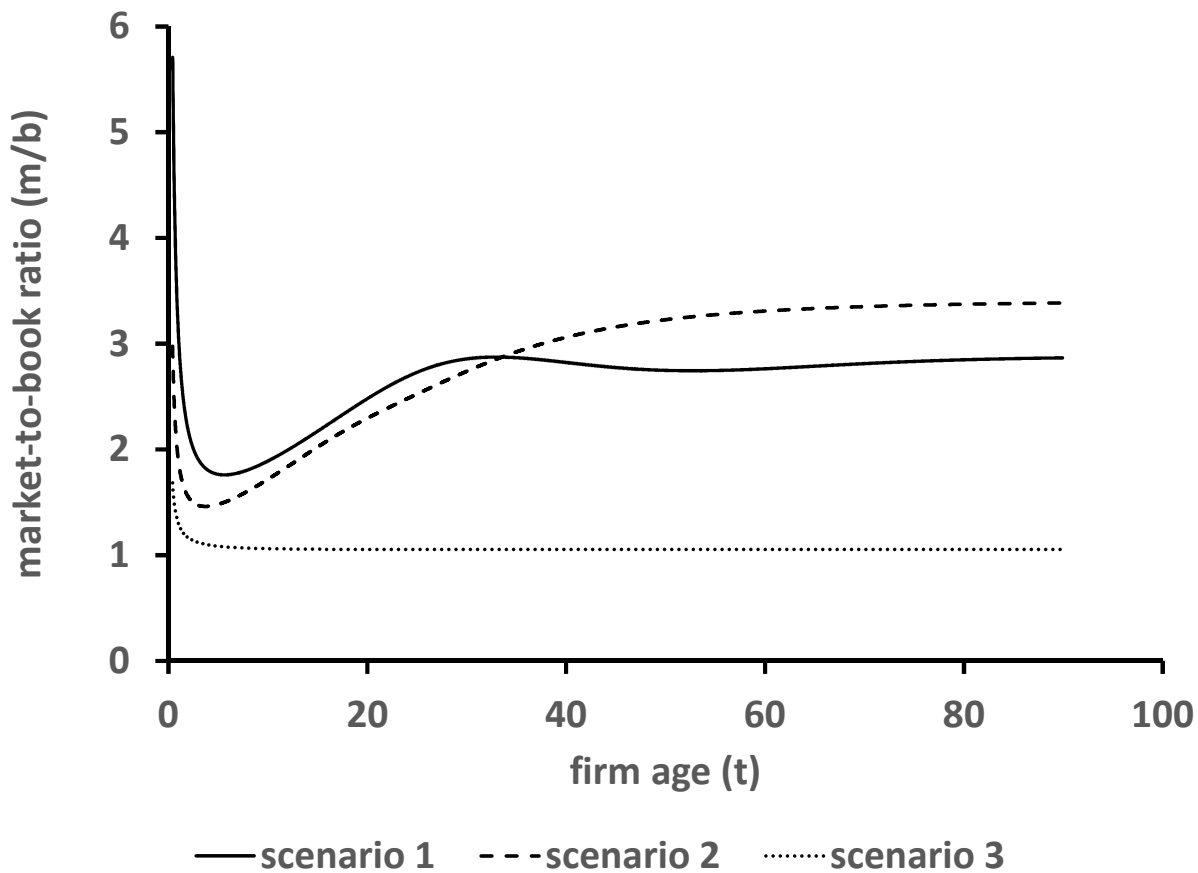


Figure 5. Histogram of estimated cost of capital values (model parameter r) for a sample of 10,805 firms with an average history of 18.6 years of financial data.

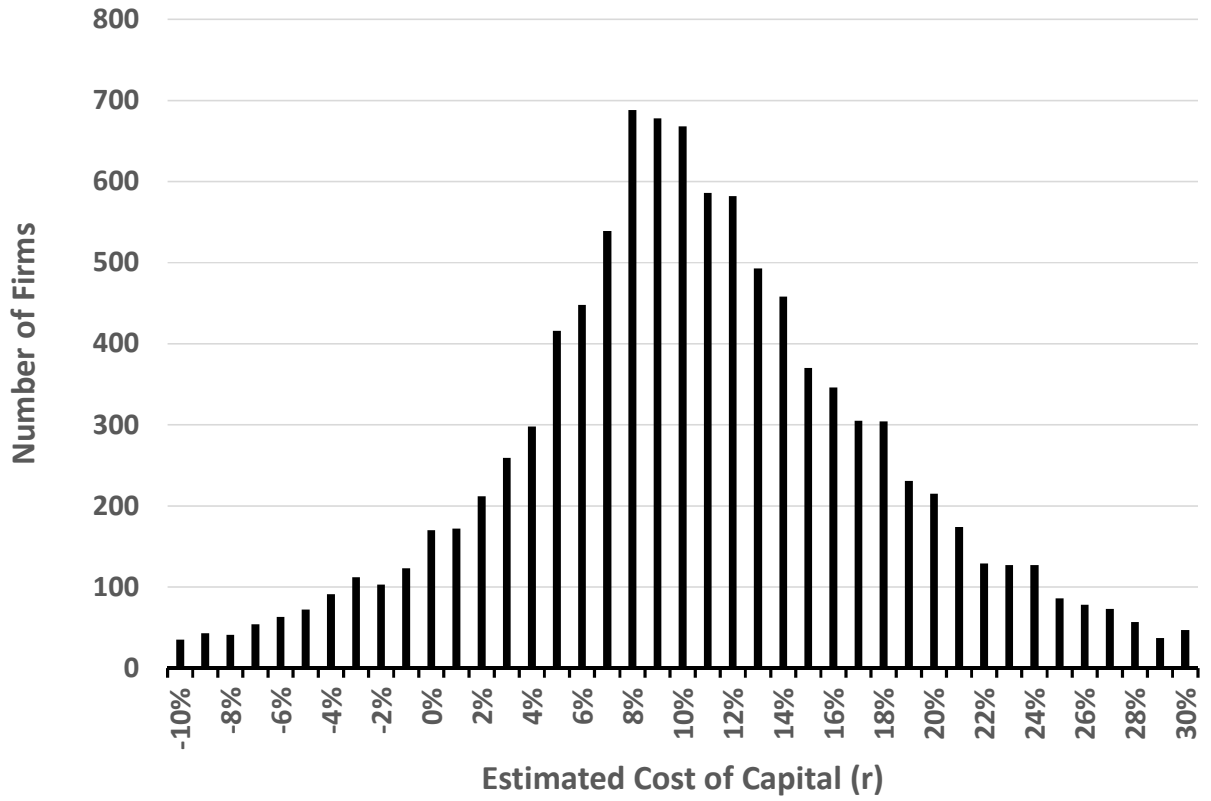


Figure 6. Scatter plot of model-implied and actual firm values. The scale is the natural logarithm of firm value in \$ millions. To compose the plot, one fiscal year was selected at random from each of the 10,805 sample firms' history, so that each data point corresponds to one fiscal year of a given sample firm.

