Real-Time Taylor Rule Ambiguities

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Abstract

We propose a general equilibrium model of the nominal and real term structure that accounts for real-time Taylor rule risks and real-time Taylor rule ambiguities. We treat Taylor rule risks and Taylor rule ambiguities as macro variables that are observed in real-time. In the estimation we find that investors are ambiguity averse with regard to GDP and inflation risk and ambiguity loving with regard to the Fed's discretionary monetary policy actions. The estimated detection error probability is 32.3% and the ambiguity loving premium for the Fed's actions is positive and strongly upward sloping for real and nominal bonds.

Keywords: Taylor rule, Multiple prior, Equilibrium, Long-run risk, Macro-finance, Term structure

JEL: E43, E44, G12

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1 Introduction

Over the last decades, the U.S. Federal Reserve (Fed) has conducted monetary policy by controlling the very short-end nominal interest rate, the Federal funds rate. Since Taylor (1993), Clarida and Gertler (1997), and Clarida, Gali, and Gertler (2000) it is widely accepted that forward looking Taylor rules provide a good tool to link variations in the Federal funds rate to variations in macroeconomic outlooks on GDP growth and inflation. Ang, Dong, and Piazzesi (2008) provide a no-arbitrage model of the nominal term structure that accounts for different Taylor rule risks: GDP risk, inflation risk, and Taylor rule implied monetary policy risk.

Our paper extends that line of research by proposing a general equilibrium model of the nominal and real term structure that accounts not only for forward-looking Taylor rule risks, but also for model uncertainty about the various forward-looking Taylor rule components. It has been overlooked in research and policy work that macro and financial economists observe in real-time the time-varying degree of model uncertainty about forward-looking GDP growth and inflation outlooks, as well as the degree of model uncertainty about Taylor rule implied monetary policy shocks. We close that gap in the literature by showing how to construct an equilibrium term structure model that accounts for real-time Taylor rule risks and real-time Taylor rule ambiguities.

Our paper provides theoretical and empirical contributions to the literature. As a theoretical contribution, we propose a recursive multiple prior utility framework in the spirit of Gilboa and Schmeidler (1989), and Chen and Epstein (2002)[1], where we account explicitly for real-time observable time-varying multiple priors on expected GDP growth, expected inflation, and the Taylor rule implied monetary policy shock. Each source of Taylor rule

[1]Other early contributions are Epstein and Wang (1994), Epstein and Schneider (2003), Chen and Epstein (2002) and Epstein and Miao (2003).

ambiguity has its own model uncertainty preference parameter and time-series behavior. The model is able to account for Taylor rule risks, ambiguity aversion, ambiguity neutrality and ambiguity love.[2]

Our paper finds several empirical novelties.

First, we propose an easy to implement method that allows the real-time extraction of the set of potential Taylor rule implied monetary policy shocks. This extends work of Ulrich (2010a) and Ulrich (2010b) who suggests an empirical method to extract in real-time the set of potential models for GDP growth and inflation. We can therefore treat Taylor rule risks, as well as Taylor rule ambiguities as observable macro state variables.

Second, while using the rich general equilibrium restrictions to estimate the term structure model with a rich panel of macro and financial data, we find that each source of Taylor rule ambiguity affects the real and nominal yield curve differently. Our results confirm Ulrich (2010a) who finds that investors are ambiguity averse with regard to the observable set of multiple inflation priors. We also confirm the empirical results of Ulrich (2010b) who finds that investors are ambiguity averse with regard to the observable set of multiple GDP growth priors. As a novelty, we find that investors are ambiguity loving with regard to the Taylor rule implied discretionary monetary policy shock.

The intuition of that finding is in accordance with the intuition of Hansen and Sargent (2007), who re-interpret a min-max robust control problem as a game between an agent and nature, where the agent maximizes her life-time utility with regard to her consumption path, while malevolent nature decides upon the law of motion for consumption. In our

[2]Klibanoff, Marinacci, and Mukerji (2005) and Ghirardato and Marinacci (2002) provide a theoretical treatment for ambiguity aversion, ambiguity neutrality and ambiguity love.

model, it is indeed the Fed which chooses the monetary policy shock, while the investor has to live with that decision. But as formulated in a 1977 amendment to the Federal Reserve Act, it is the goal of U.S. monetary policy to promote maximum sustainable output and employment, while promoting stable prices. To fulfill the first goal, the Fed often deviates negatively from the systematic forward-looking Taylor rule if aggregate demand weakens and if a thread of a recession intensifies. Our finding suggests that although the Fed creates ambiguity by not committing explicitly to a specific Taylor rule, its benevolent discretionary monetary policy actions are regarded as benevolent instead of malevolent. That means that the Fed's goal to stabilize output through discretionary monetary policy shocks is appreciated by the financial market. In technical terms that means that investors are ambiguity loving with regard to the Taylor rule implied discretionary monetary policy shocks.

Third, model uncertainty aversion with regard to forward-looking GDP growth and forward-looking inflation leads the investor to act and do pricing as if the path of future GDP followed a worst-case paths. The model uncertainty loving of the discretionary Fed actions motivates the investor to act and do pricing as if the path of future GDP followed a best-case path. Both channels balance each other. The overall worst-case GDP and inflation path is not distinguishable from the investors empirical reference model. We confirm this visually and formally with an estimated detection error probability of 32.3%. The last number indicates that even after observing our rich panel of macro and financial data, if the investor had to choose between the worst-case and the reference model, she would choose the wrong model with a probability of 32.3%.

Forth, our estimates show that the ambiguity loving premium for the discretionary monetary policy shock produces a strongly upward sloping nominal yield curve, while the long-run GDP (inflation) ambiguity premium produces a negative (positive) and slightly

downward (upward) sloping term premium. Consistent with Ulrich (2010a) we find that the long-run GDP ambiguity premium is negative because real bonds are a natural hedge against long-run GDP risk ambiguity. The positive slope that long-run inflation ambiguity adds to the nominal yield curve is lower than the reduction through long-run GDP ambiguity. The overall long-run ambiguity premium in nominal yields is therefore slightly downward sloping.

Fifth, the estimated ambiguity premium for the discretionary monetary policy shock is 35 basis points for the two year nominal yield and 140 basis points for the nine year yield. In contrast, the overall long-run ambiguity premium is 80 basis points for the two year yield and 55 basis points for the nine year yield. The resulting overall ambiguity premium is upward sloping, 115 basis points for the two year yield and 195 basis points for the nine year yield.

Sixth, our finding shows that a macro model which accounts for the empirically observed set of potential discretionary Fed actions can very easily recover the upward sloping real yield curve. That is an astonishing and very important finding, because recent research have pointed out the difficulty that parsimonious general equilibrium models have in proposing a mechanism for a positive real term premium [Piazzesi and Schneider (2006), Ulrich (2010b)]. The economic mechanism in our parsimonious macro model is very intuitive. The investor is confronted with a set of potential Taylor rule implied monetary policy shocks. She does not know which monetary shock is chosen by the Fed. The estimated preference parameter for that kind of ambiguity reveals that the investor loves that kind of model uncertainty, because she trusts that the Fed acts as a benevolent player. The c.p. worst-case GDP growth process becomes actually a best-case GDP growth process. The investor prices therefore real and nominal bonds with a c.p. slightly higher GDP growth rate. The corresponding real and nominal bond prices are lower which means the

resulting yields are higher than in a world where the Fed commits to a Taylor rule implied monetary policy shock.

Seventh, in the time-series we find that the Fed ambiguity premium has peaked at 2 percent for the ten-year nominal and real yield in the early 1980s. During the Great Moderation, that premium stabilized around 1.2%. It increased during the current financial market crisis from 1.2% to 1.5%. We find that the Fed ambiguity premium affects real bond yields and nominal bond yields in the same way. That implies that break-even inflation rates are not affected by that premium. That is in sharp contrast to the inflation ambiguity premium, which affects nominal yields much stronger than it affects real bond yields. That is consistent with Ulrich (2010a).

In summary, we find that a tiny, not detectable amount of model uncertainty about all components of the Taylor rule generates interesting and plausible empirical regularities in the cross-section and time-series of U.S. government bonds.

2 Model

2.1 Domain

Time is continuous and varies over $t \in [0, ..., \infty)$. Real and nominal macroeconomic risk is represented by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q^0)$, which satisfies the usual conditions. The probability measure Q^0 stands for the reference macroeconomic model for the economy. For intuitive reasons it is useful to treat the solution of the reference model under Q^0 as the solution to the rational expectations model. All expectations in the reference model are taken under Q^0. We denote these expectations as $E[.]$ instead of $E^{Q^0}[.]$. The probability measure for the robust economy will be determined endogenously in equilibrium. We will denote that measure as Q. We assume all processes lie within a

well defined space and are adapted to the underlying filtration.

2.2 Policy Rules

We begin by describing a standard Taylor (1993) policy rule compared to a forward looking Taylor rule as in Clarida and Gertler (1997), Clarida, Gali, and Gertler (2000), and Ang, Dong, and Piazzesi (2008). Next, we show that our Taylor rule extends the existing Taylor rules by incorporating the empirically observed set of multiple priors on next quarters expected inflation, expected GDP growth and next quarter expected short-term interest rate. We derive a term structure model that takes these different sources of model uncertainty into account.

2.2.1 Policy Rule with Output Gap and Inflation

The standard Taylor (1993) rule imposes that monetary authority sets the current nominal short rate as an affine function of inflation and the output gap:

$$R_t = c + ag_t + b\pi_t + f_t^{std} \tag{2.1}$$

where R_t is the value of the Federal funds rate (FFR), g_t is the output gap, and π_t is inflation. Such a Taylor rule argues that the systematic part of the Fed's interest rate policy can be described by $c + ag_t + b\pi_t$ whereas the mean-zero residual f_t^{std} can be interpreted as the Fed's discretionary monetary policy shock.

2.2.2 Policy Rule with Expected Output Gap and Expected Inflation

Clarida and Gertler (1997) and Clarida, Gali, and Gertler (2000) suggest a forward looking Taylor rule where the Fed sets the FFR based on the expected output gap and expected inflation over the next few quarters. For example, a forward-looking Taylor rule that uses expected output gap and expected inflation over the next quarter takes the form:

$$R_t = c + aE_t[g_{t+1}] + bE_t[\pi_{t+1}] + f_t^F, \tag{2.2}$$

where f_t^F is the forward looking Taylor rule monetary policy shock.

Previous empirical research in term structure modeling and Taylor rule modeling has found that the expected inflation component drives more than 90% in variations in R_t and longer dated nominal interest rates. **?**) and Ang, Bekaert, and Wei (2008) show that most of the variation in short-term and long-term nominal bond yields can be explained by variations in expected inflation.

Recent research on macroeconomic model uncertainty has pointed out that investors are confronted with a set of multiple GDP and inflation models [**?**), **?**)]. **?**) uses the observable set of different GDP and inflation risk forecasts that are published by the Survey of Professional Forecasters (SPF) as a proxy for the set of potential long-run macro risk models. We extend and combine that idea with the insights of forward looking Taylor rules as in Clarida and Gertler (1997), Clarida, Gali, and Gertler (2000) and Ang, Dong, and Piazzesi (2008) to account for model misspecification doubts about the Fed's discretionary monetary policy shock.

2.2.3 Forward Looking Taylor Rule with Forward Looking Long-Run Macro Risk

Previous research assumes that f_t^F is observed, or that the ex-post estimate coincides with the ex-ante expected shock.[3] We extend that line of research in several dimensions. First, we use not only a forward looking model for long-run macro risk ($E_t[g_{t+1}]$ and $E_t[\pi_{t+1}]$), but we also use the forward looking estimate of the nominal short-term interest rate itself, i.e. $E_t[R_{t+1}]$. The advantage of that extension is that it allows the incorporation of the

[3]Ex-post estimates of f_t^F are done in Ang, Dong, and Piazzesi (2008) and Ang, Boivin, Dong, and Loo-Kung (2009).

rich forward looking data set that the SPF provides on forward-looking long-run macro risk and the forward-looking short-term nominal interest rate. Second, we incorporate the empirical evidence that different SPF macro-econometric specialists forecast different long-run macro risk ($E_t[g_{t+1}]$ and $E_t[\pi_{t+1}]$) and different short-term nominal interest rates $E_t[R_{t+1}]$. The advantage of that extension is, it allows the extraction of the set of potential Taylor rule models in real-time.

Our suggested forward looking Taylor rule uses expected one-quarter ahead GDP growth, $E_t[\Delta \ln Y_{t+1}]$ where Y stands for aggregate GDP, instead of the one-quarter ahead expected output gap. The main reason for that modeling choice is that the SPF data set publishes only forecasts on the former.[4] Our forward-looking Taylor rule is:

$$E_t[R_{t+1}] = c + aE_t[\Delta \ln Y_{t+1}] + bE_t[\pi_{t+1}] + E_t[f_{t+1}^{MP}], \tag{2.3}$$

where $E_t[f_{t+1}^{MP}]$ is the anticipated realization of the discretionary monetary policy shock in next period. The SPF data base allows us to track the forecasts on $E_t[R_{t+1}]$, $E_t[\Delta \ln Y_{t+1}]$, and $E_t[\pi_{t+1}]$ for different macro-econometric specialists. This leaves the econometrician in every point in time t with a set of potential long-run risk models $E_t[\Delta \ln Y_{t+1}]$, and $E_t[\pi_{t+1}]$ and with a set of potential Fed instruments $E_t[R_{t+1}]$. We assume that for every time period t, different forecasters use the same Taylor weights a and b, but implicitly anticipate a different realization of the discretionary monetary policy shock $E_t[f_{t+1}^{MP}]$. This allows investors and econometricians to extract an observable set of potential real-time $E_t[f_{t+1}^{MP}]$ that different SPF forecasters regard as plausible. This procedure provides a very natural way to account for model misspecification doubts about the Taylor Rule.

In more detail, for every $t \in [1, ..., T]$ in the sample, we run a cross-section regression

[4] As a robustness check we check the correlation between the output gap and the forecast on the one-quarter ahead expected GDP growth rate. Both are highly negatively correlated, suggesting that both statistics carry similar information about the real economy.

of equation (2.3) to extract a, b and the implied cross-section of $E_t[f_{t+1}^{MP}]$ forecasts. We follow **?**) and use a multiple times the variance of the cross-sectional $E_t[f_{t+1}^{MP}]$ to characterize the size of the set of potential discretionary monetary policy shocks. While the estimated variance provides an observable macro time-series of the set of potential Taylor rule models, we will estimate the multiple as part of the Taylor rule ambiguity premium.

2.3 Asset Pricing with Taylor Rule Ambiguities

2.3.1 Macro Risk

When working with a Taylor rule of type (2.3), one is confronted with three sets of multiple priors, a set of potential GDP growth models, a set of potential inflation models and a set of potential discretionary monetary policy shocks. **?**) studies term structure implications of a general equilibrium model with the first two sets of multiple priors. For ease of comparison, we follow his notation and we focus in our analysis on the real-time ambiguity of discretionary monetary policy shocks. The GDP growth process, $d\ln Y$, and inflation, $d\ln p$, of **?**) follow[5]:

$$d\ln Y_t = (g_0 + z_t)dt + \sqrt{\sigma_{0g} + \sigma_{1g}u_t}dW_t^Y, \tag{2.4}$$

$$d\ln p_t = (p_0 + w_t)dt + \sqrt{\sigma_{0g} + \sigma_{1g}u_t}\rho_{pg}dW^Y + \sqrt{\sigma_{0p} + \sigma_{1p}v_t}dW^p, \tag{2.5}$$

where the Brownian motions are pairwise orthogonal. Although our model does not rely on that notion, we point out that the long-run risk literature would call z, the long-run GDP risk component, and w as the long-run inflation component.

We put the state variables that drive macro risk into a vector $\bar{X}$, where $\bar{X}_t = [u_t\, v_t\, w_t\, z_t]'$. We assume it follows a continuous-time $AR(1)$ process with pairwise orthogonal innova-

[5]We work with an endowment economy and with a representative agent. In these models, GDP and consumption coincide in equilibrium. We prefer to work with GDP directly.

tions. Under the reference measure that means

$$d\bar{X} = \kappa \bar{X} dt + \Sigma dW, \tag{2.6}$$

where κ is a diagonal 4×1 matrix which captures the $AR(1)$ nature of macroeconomic risk, Σ is a 4×5 volatility matrix and the 5×1 vector dW collects the pairwise orthogonal innovations to macroeconomic risk.

We extend the observable long-run ambiguity set-up of **?**) to account for real-time model misspecification doubts about the discretionary monetary policy shock. The extension is beneficial, because the former author finds a significant premium for long-run ambiguity and we want to control and account for that. Extending that framework ensures that model misspecification doubts about the discretionary monetary policy shock does not simply substitute for long-run ambiguity but instead acts as an independent additional macro factor. In addition, the extension accounts for the empirical fact that the entire forward-looking Taylor rule in (2.3) is ambiguous. This holds because the first two components of the rhs of that Taylor rule coincide with long-run GDP risk and long-run inflation risk, whereas the last component coincides with the forward-looking monetary policy shock. Our model extension incorporates the observed set of potential models for these components separately.

The corresponding volatility matrix takes the form

$$\Sigma dW := \begin{pmatrix} \sigma_u & 0 & 0 & 0 & 0 \\ 0 & \sigma_v & 0 & 0 & 0 \\ 0 & 0 & \sigma_w & 0 & 0 \\ 0 & 0 & \sigma_{2z} & \sigma_{1z} & \sigma_{3z} \end{pmatrix} \begin{pmatrix} dW^u \\ dW^v \\ dW^w \\ dW^z \\ dW^f \end{pmatrix}. \tag{2.7}$$

In the data, σ_{2z} is negative, while σ_{3z} is positive. The latter indicates a positive empirical relationship between short-run fluctuations in the discretionary monetary policy shock

and short-run fluctuations in long-run GDP risk. Such a positive correlation is intuitive because we observe in reality that the Fed tends to lower (increase) the Federal funds rate if the economy is hit by a negative (positive) long-run GDP growth shock.[6]

With regard to the real-time ambiguity about the discretionary monetary policy shock: First, our model argues that the best guess (empirical reference model) of the investor is that changes in the Fed's monetary policy shock do not systematically affect long-run GDP risk. Under that reference specification, such a shock has only an unsystematic but positive impact on expected GDP growth, i.e. $\sigma_{3z} > 0$. Second, the agent is able to extract the empirically observed set of potential discretionary monetary policy shocks. Intuitively, this ambiguity means that the investor does not fully trust that the "true" mean of the discretionary monetary policy shock is zero, indeed. If there was not model uncertainty, the investor would know that the Fed draws that monetary policy shock from a zero mean Gaussian distribution.

2.3.2 Real-Time Taylor Rule Ambiguities

The investor wants a robust decision rule for her consumption and investment path so that she does well, in terms of utility, across all potentially correct Taylor rule models. She obtains robustness by determining endogenously (via a min-max multiple prior control problem) the optimal amount of Taylor rule misspecification that she should take into account. That optimal amount is called $h = (h_f, h_w, h_z)$ and accounts for the three dimensions of the Taylor rule ambiguity. Note that h is known to the investor in every point in time and in every state of the world.

[6] The monetary policy shock dW^f is extracted from running a regression of the Taylor rule (2.3) with the median forecasts as the corresponding values for $E_t[\Delta \ln Y_{t+1}]$ and $E_t[\pi_{t+1}]$ and the realized Federal Funds Rate. In an empirical analysis, Ang, Bekaert, and Wei (2007) could not find a single model that beats the out-of-sample forecasting performance of the median inflation forecast.

The endogenous amount of Taylor rule ambiguity coincides with the market price of ambiguity in the multiple prior literature or with the market price of one unit model uncertainty in the robust control literature.[7] We derive the Taylor rule ambiguity premiums in the next section.

According to equation (2.6), we state the long-run GDP risk model under the robust probability measure Q:

$$dz = \kappa_z z dt + \sigma_{1z}(dW^{z,h} + h_z dt) + \sigma_{2z}(dW^{w,h} + h_w dt) + \sigma_{3z}(dW^{f,h} + h_f dt). \tag{2.8}$$

The last equation together with equation (2.6) states than an ambiguity averse investor that is confronted with a set of multiple long-run risk models and multiple models for the discretionary monetary policy shock does not fully trust her reference model in equation (2.6). Instead, she adjusts the shocks to long-run risk and to the Taylor rule implied monetary policy shock by the corresponding market price of ambiguity. It coincides with h_z (h_w) for long-run GDP (inflation) ambiguity and with h_f for the Fed ambiguity.

The long-run inflation risk model under the robust probability measure Q follows analogously:

$$dw = \kappa_w w dt + \sigma_w(dW^{w,h} + h_w dt). \tag{2.9}$$

Instead of treating the set of potential models as unobservable and latent, we follow Ulrich (2010a) and Ulrich (2010b) and characterize that set by the in real-time observed amount of model estimation risk:

$$UB_z(t) := A_z \eta_z^2(t) \qquad UB_w(t) := A_w \eta_w^2(t) \qquad UB_f(t) := A_f \eta_f^2(t), \tag{2.10}$$

[7] Compare Epstein and Wang (1994), Epstein and Schneider (2003), Chen and Epstein (2002), Epstein and Miao (2003) for the former and Anderson, Hansen, and Sargent (2003), Cagetti, Hansen, Sargent, and Williams (2002), Hansen and Sargent (2007), Hansen and Sargent (2005), Hansen, Sargent, Turmuhambetova, and Williams (2005), Maenhout (2004), and Maenhout (2006) for the latter.

where the time-varying components capture the observable amount of model estimation risk. The constants A are weakly positive scaling parameters. We think of this amount of model estimation risk as a confidence interval that surrounds the reference model. The scaling parameter captures for how much of the confidence interval the investor wants model misspecification protection. For parsimonious reasons we assume that the observable amount of model estimation risk follow a pairwise orthogonal continuous-time $AR(1)$ process.

2.4 Preferences

We assume a log utility investor who faces risk and uncertainty about her dividend stream. Such an investor is a min-max optimizer.[8] The investor lives in an endowment economy which leaves her only with the minimization problem. The dynamic minimization problem at hand is:

$$\min_{Z \in Z(UB)} E^Z \left[\int_t^\infty e^{-\rho(s-t)} \ln Y_s ds | \mathcal{F}_t \right] \tag{2.11}$$

$$s.t. \tag{2.12}$$

$$d \ln Y_t = (g_0 + z_t)dt + \sqrt{\sigma_{0g} + \sigma_{1g} u_t} dW_t^Y \tag{2.13}$$

$$dz_t = \kappa_z z_t dt + \begin{pmatrix} \sigma_{1z} & \sigma_{2z} & \sigma_{3z} \end{pmatrix} \begin{pmatrix} dW^{z,h} + h_z dt \\ dW^{w,h} + h_w dt \\ dW^{f,h} + h_f dt \end{pmatrix} \tag{2.14}$$

$$Z(UB) = \{Z \in \mathcal{Z} : \frac{1}{2} h_z^2(t) \leq UB_z(t)\} \cap \{Z \in \mathcal{Z} : \frac{1}{2} h_w^2(t) \leq UB_w(t)\}$$
$$\cap \{Z \in \mathcal{Z} : \frac{1}{2} h_f^2(t) \leq UB_f(t)\}, \tag{2.15}$$

where ρ is the investors' subjective discount factor. The last equation states the relative entropy constraints which restricts the amount of models that the investor treats as po-

[8] Compare Hansen and Sargent (2007) for an excellent treatment of min-max and robustness.

tentially correct. The solution to the problem is summarized in the next proposition.

Proposition 1 (Premiums for Taylor Rule Ambiguities) *For a log utility investor, who is confronted with a set of multiple priors on expected GDP growth, expected inflation and the discretionary monetary policy shock, it is optimal to distort the corresponding shocks by the relevant ambiguity premiums* $h = (h_z, h_w, h_f)$*:*

$$h_z(t) = -\sqrt{2 \cdot UB_z(t)}, \quad h_w(t) = \sqrt{2 \cdot UB_w(t)}, \quad h_f(t) = \pm\sqrt{2 \cdot UB_f(t)}, \tag{2.16}$$

where UB_z, UB_w, UB_f *coincide with the real-time observations of the upper boundaries of the corresponding set of multiple priors.*

The proposition has several nice interpretations. First, the first two equations in (2.16) state that the absolute size of the long-run GDP and inflation ambiguity premium increase monotonically with the amount of its potential models. In ambiguous times, where the set of potential long-run risk models increases, the investor *reduces* (*increases*) her expected prospects for the real growth rate of GDP (inflation) by less than one to one.[9] Consistent with our theoretical findings, Ulrich (2010b) finds a positive long-run inflation ambiguity premium in nominal bond yields and a negative long-run GDP ambiguity premium.

Second, with the third equation we allow explicitly for a positive and a negative ambiguity premium for discretionary monetary policy shocks. The reason for that novelty is that we do not want to artificially restrict the model to account for ambiguity aversion with regard to that shock. Intuitively, $h_f(t) = +\sqrt{2 \cdot UB_f(t)}$, means that investors are ambiguity loving with regard to discretionary monetary policy shocks. Klibanoff, Marinacci, and Mukerji (2005) and Ghirardato and Marinacci (2002) provide a theoretical treatment for ambiguity love. The concept of ambiguity love captures the opposite notion of ambiguity aversion. Instead of searching for a decision rule that lets the ambiguity

[9] This concave relationship arises because shocks to long-run GDP risk are Gaussian.

averse agent act as if consumption follows a worst-case path, ambiguity loving agents acts as if their consumption path follows a best case path. It is intuitively appealing that different random sources of ambiguity might generate either ambiguity aversion or ambiguity loving.[10] To the best of our knowledge, there does not exist a single paper that analysis theoretically and empirically how an investor prices assets if confronted with a set of multiple models for the discretionary monetary policy shock.

Third, it is intuitive, why ambiguity averse investors might indeed love ambiguity with regard to the Fed's discretionary policy shock. For consumption uncertainty, Hansen and Sargent (2007) provide a very nice intuition of why one can think of a min-max optimization problem as a game between an agent and nature, where the agent maximizes her life-time utility with regard to her consumption path, while malevolent nature decides upon the law of motion for consumption. Applying that intuition to the empirically observed set of multiple discretionary monetary policy shocks suggests that nature in that example is the Federal Reserve. Although we do not take a stand in equation (2.16) of whether the investor is ambiguity averse or ambiguity loving with regard to that shock, one can easily defend the argument that the Fed is not the malevolent nature in that game, but rather benevolent nature. It is indeed the Fed which chooses the monetary policy shock. But as formulated in a 1977 amendment to the Federal Reserve Act, it is the goal of U.S. monetary policy to promote maximum sustainable output and employment, while promoting stable prices. To fulfill the first goal, the Fed often deviates negatively

[10] Ulrich (2010a) finds that investors are ambiguity averse when confronted with an observable set of multiple inflation priors. Ulrich (2010b) finds that investors are ambiguity averse when confronted with an observable set of multiple long-run macro risk models. Epstein and Wang (1994), Epstein and Schneider (2003), Chen and Epstein (2002), Epstein and Miao (2003) and Anderson, Hansen, and Sargent (2003), Cagetti, Hansen, Sargent, and Williams (2002), Hansen and Sargent (2007), Hansen and Sargent (2005), Hansen, Sargent, Turmuhambetova, and Williams (2005), Maenhout (2004), and Maenhout (2006) work with ambiguity aversion with regard to consumption uncertainty.

from the systematic Taylor rule if aggregate demand weakens and if a thread of a recession intensifies.

So, although the Fed does not commit to an explicit Taylor rule, its benevolent actions in order to stabilize output makes it quite likely that ambiguity averse investors love the the ambiguity about the Fed's discretionary shock instead of being averse towards it. A further indication for that intuition is that investors called the conduct of monetary policy during the Allan Greenspan area, as "Greenspan Put". That expression has been chosen to symbolize that investors strongly believed that the Fed under Greenspan will lower the Federal funds rate significantly in times of crisis (low consumption growth) to increase the expected consumption path of investors.[11] Intuitively, that might also motivate the notion of ambiguity loving investors who search for the best-case consumption path given an observable set of potential models for the Fed's discretionary monetary policy shock.

Fourth, we do not take a stand on whether the investor is ambiguity averse, ambiguity neutral or ambiguity loving with regard to the set of multiple monetary policy shocks. We let the data determine that.

One can re-write the equilibrium outcome as:

$$h_f(t) = \pm m_f \eta_f(t), \quad h_z(t) = m_z \eta_z(t), \quad h_w(t) = m_w \eta_w(t), \tag{2.17}$$

where m_f equals $m_f := \sqrt{2A_f} \in R^+$, m_z equals $m_z := -\sqrt{2A_z} \in R^-$ and m_w equals $m_w := \sqrt{2A_w} \in R^+$. As Ulrich (2010c) explains, from an asset pricing perspective one can regard m_f, m_z, m_w as the model uncertainty counterpart to the coefficient of relative risk aversion. Intuitively, this makes sense because if these parameters are zero the investor requires no compensation for model uncertainty, regardless of how large the actual amount

[11] The Fed has reacted to the current financial crisis with similar measures under the current Fed chairman Ben Bernanke.

of observed model estimation risk. On the other hand, a gigantic value of m corresponds to a gigantic uncertainty premium even for a very small amount of model estimation risk. Moreover, the intuition of Ulrich (2010a) and Ulrich (2010b) carries over to the market price of model uncertainty about potential Fed actions. The higher the amount of model estimation risk in that variable, the more uncertain is the investor about the future Fed action. For an ambiguity averse investor it becomes optimal to require a model uncertainty premium that increases linearly on the amount of model estimation risk.

2.5 The Fed and the Bond Market

In the appendix we derive formally, that continuously compounded real reals, y^r, and continuously compounded nominal bond yields, $y^\$$, follow an affine relation:

$$y_t^r(\tau) = \alpha^r(\tau) + \beta^{r\,\prime}(\tau)S(t),\ S(t) = (u(t)\, z(t)\, h_f(t)\, h_w(t)\, h_z(t))', \tag{2.18}$$

$$y_t^\$(\tau) = \alpha^\$(\tau) + \beta^{\$\,\prime}(\tau)\bar{S}(t),\ \bar{S}(t) = (S(t)\, w(t)\, v(t)), \tag{2.19}$$

where A^r, $A^\$$ and B^r, $B^\$$ are deterministic functions of the underlying economy.

3 Data and Econometric Methodology

The goal of this section is to briefly describe the data and the econometric methodology used for estimating the equilibrium model. The empirical exercise draws upon Ulrich (2010a) and Ulrich (2010b). In this analysis we focus on ambiguity with regard to the Taylor rule. We include the insights of long-run ambiguity from Ulrich (2010b), so that we explicitly discriminate the Taylor rule ambiguity from potential long-run macro risk ambiguity.

3.1 Data

Characterizing to our Taylor Rule asset pricing model is that the investor observes a set of potentially correct models for real and nominal macro growth as well as for potential Fed actions. We characterize the set of potential macro models with the Survey of Professional Forecasters (SPF). Ulrich (2010a) introduced this idea and it was generalized to real and nominal macro growth by Ulrich (2010b). Ang, Bekaert, and Wei (2007) study out-of-sample inflation predictions and could not find a model which reliably beats the the median SPF forecast. We therefore assume that z coincides with the demeaned median forecast of GDP growth, g_0 coincides with its unconditional expectation and w, coincides with the demeaned forecast for inflation and p_0 coincides with its unconditional expectation.

We match the amount of model estimation risk with the dispersion of SPF forecasts. Furthermore, we incorporate several macro variables and bond yields into the estimation. As macro variables we use realized GDP growth, realized inflation and the Federal Funds rate. That data is provided by the St. Louis FRED data base. For bond yields we use continuously compounded nominal U.S. government bond yields of maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 years. We also use TIPS as a proxy for real yields. The maturities that we use are 5, 6, 7, 8, 9, and 10 years. Except TIPS data, all data spans the period third quarter 1981 to the second quarter of 2009. The TIPS yields span the third quarter of 1981 until the second quarter of 2009. We estimate the model by Maximum Likelihood and invert the latent volatility states by inverting the affine yield relation for the one year and the ten year nominal bond yield.

3.2 Estimation and Empirical Identification of Taylor Rule Ambiguities

We identify the amount of model uncertainty in expected GDP growth and expected inflation as in Ulrich (2010b). As a novelty, we propose a new method for estimating the amount of model estimation risk in the Fed's policy action. We determine the empirically observed set of potential Taylor rule models based on our Taylor rule in equation (2.3) and based on the SPF data. The SPF data base allows us to track the forecasts on $E_t[R_{t+1}]$, $E_t[\Delta \ln Y_{t+1}]$, and $E_t[\pi_{t+1}]$ for each macro-econometric specialist. This leaves the econometrician in every point in time t with a set of potential long-run risk models $E_t[\Delta \ln Y_{t+1}]$, and $E_t[\pi_{t+1}]$ and with a set of potential Fed instruments $E_t[R_{t+1}]$. We assume that for every time period t, different forecasters use the same Taylor weights a and b, but implicitly anticipate a different realization of the forward-looking Taylor rule implied monetary policy shock $E_t[f_{t+1}^{MP}]$. We extract this anticipated shock through a cross-sectional regression. We use a constant times the dispersion in $E_t[f_{t+1}^{MP}]$ to characterize the size of the set of potential forward-looking Taylor rule monetary policy shocks.

4 Empirical Results

4.1 Ambiguity Loving of Fed Actions explains Positive Term Premium in Tips and Nominal Bonds

The unconstrained estimation of the market price of model uncertainty about the discretionary monetary policy shock is positive, i.e. $m_f = 21.3$. This finding suggests that investors belief that the Fed uses its discretionary power over the Taylor rule to support the growth of the aggregate GDP path. As a result, investors confidence in the Fed leads them to act in the economy as if the underlying GDP growth process grew stronger than

it actually does. The inclusion of ambiguity aversion with regard to the observable set of multiple long-run risk models leads the investor to reduce her growth prospects of the real economy. This counter acts with the increased growth prospects that are induced by ambiguity loving of the discretionary monetary policy shock. Figure 6 shows: First, the worst-case model for GDP growth is very close to the (empirical) reference model. Second, only in some quarters during the early and mid 1980s did investors's worst-case GDP growth model exceed the (empirical) reference counterpart. For all other periods, we find that ambiguity aversion with regard to long-run macro risk dominates the anticipated short-run macro dynamics compared to ambiguity loving of the discretionary monetary policy shock.

The previous result does not imply that long-run macro ambiguity aversion dominates the total ambiguity premium. In fact, we find that the opposite is true. The right panel of Figure 3 and Figure 1 show that the ambiguity loving premium for the discretionary monetary policy shock produces a strongly upward sloping nominal yield curve, while the long-run GDP (inflation) ambiguity premium produces a negative (positive) and slightly downward (upward) sloping term premium. Consistent with Ulrich (2010a) we find that the long-run GDP ambiguity premium is negative because real bonds are a natural hedge against long-run GDP risk ambiguity. The positive slope that long-run inflation ambiguity adds to the nominal yield curve is lower than the reduction through long-run GDP ambiguity. The overall long-run ambiguity premium in nominal yields is therefore slightly downward sloping. Table 1 explains that the empirically observed set of potential discretionary Fed models is much more persistent than the long-run ambiguity counterpart. It is nine times more persistent than the long-run GDP ambiguity premium. This gives the Fed ambiguity premium a greater impact for long-term assets and produces an upward sloping nominal yield curve.

The upper panel of Figure 1 shows that the ambiguity premium for the discretionary monetary policy shock is 35 basis points for the two year nominal yield and 140 basis points for the nine year yield. In contrast, the overall long-run ambiguity premium is 80 basis points for the two year yield and 55 basis points for the nine year yield. The resulting overall ambiguity premium is upward sloping, 115 basis points for the two year yield and 195 basis points for the nine year yield.

Figure 2 and the right panel of Figure 4 show our finding that a macro model with Fed ambiguity can very easily recover the upward sloping real yield curve. That is an astonishing and very important finding, because recent research has pointed out the difficulty that parsimonious general equilibrium models have in proposing a mechanism for a positive real term premium [Piazzesi and Schneider (2006), **?**)]. The economic mechanism in our parsimonious macro model is very intuitive. The investor is confronted with a set of potential Taylor rule implied monetary policy shocks. She does not know which monetary shock is chosen by the Fed. The estimated preference parameter for that kind of ambiguity reveals that the investor loves that kind of model uncertainty. The c.p. worst-case GDP growth process becomes actually a best-case GDP growth process. The investor prices therefore real and nominal bonds with a c.p. slightly higher GDP growth rate. The corresponding real and nominal bond prices are lower which means the resulting yields are higher than in a world where the Fed commits to a Taylor rule implied monetary policy shock. The difference in both growth rates is very tiny, as one can see from the estimated detection error probability of 32.3% and from Figure 5 and Figure 6. Besides this hard to detect statistical difference between the reference and the worst-case model, the positive ambiguity loving Fed premium leads to a sizable ambiguity premium in Tips and nominal yields.[12]

[12]The determination of the detection error probabilities follows directly from Anderson, Hansen, and Sargent (2003) and Maenhout (2006).

Figure 7 to 10 confirms that argument. The Fed ambiguity premium has peaked at 2 percent for the ten-year nominal and real yield in the early 1980s. During the Great Moderation, that premium stabilized around 1.2%. It increased during the current financial market crisis from 1.2% to 1.5%. We find that the Fed ambiguity premium affects real bond yields and nominal bond yields in the same way. That implies that break-even inflation rates are not affected by that premium. That is in sharp contrast to the inflation ambiguity premium, which affects nominal yields much stronger than it affects real bond yields. That is consistent with Ulrich (2010a) and Ulrich (2010b).

The left panel of Figure 3 decomposes the nominal yield curve into its components. That decomposition shows that the stochastic inflation risk premium is basically zero for all maturities. The term structure of inflation expectations (empirical measure) is downward sloping, 3 percent for a two year horizon and 2 percent for a nine year horizon. The upward sloping nominal yield curve inherits the positive slope from the real yield curve and from the ambiguity premium. The right panel of Figure 3 and both panels of Figure 4 show that the upward sloping real yield curve as well as the upward sloping ambiguity premium inherits its shape from the upward sloping ambiguity premium for the discretionary monetary policy shock.

5 Conclusion

This paper derives a general equilibrium term structure model which accounts for Taylor rule risks and Taylor rule ambiguities. We estimate the model and find while investors are ambiguity averse model misspecification doubts about GDP growth and inflation, they are ambiguity loving about the Taylor rule implied discretionary Fed shocks. That "ambiguity love" results very naturally in an upward sloping real and nominal yield curve. That finding is therefore not only the first empirical account of ambiguity loving behavior, but

it also resolves a long-standing puzzle of why TIPS and nominal bond yields are upward sloping.

The intuitive reason for ambiguity love with regard to discretionary Fed actions is that investors appreciate that the Fed deviates from the Taylor rule in order to boost aggregate GDP. This implies that although investors do not know how the Fed will act, they strongly believe in the goal of the Fed to smooth negative GDP fluctuations through lowering the Federal funds rate lower than suggested by standard Taylor rules.

Taking all theoretical and empirical results together, we conclude that the observed set of multiple priors on discretionary Fed actions is very important in recovering an upward sloping real and nominal yield curve in a parsimonious macro model. The analysis suggests that it is very fruitful for future research as well as policy making to take not only Taylor rule risks into account, but also to be aware of the consequences of Taylor rule ambiguities.

A Appendix

A.1 Recursive Multiple-Prior Optimization Problem

The dynamic minimization problem is given by

$$\min_{Z\in Z(UB)} E^Z\left[\int_t^\infty e^{-\rho(s-t)}\ln Y_s ds|\mathcal{F}_t\right] \tag{A.1}$$

$$s.t. \tag{A.2}$$

$$d\ln Y_t = (g_0+g_1 z_t)dt + \sqrt{\sigma_{0g}+\sigma_{1g}u_t}dW_t^Y \tag{A.3}$$

$$dz_t = \kappa_z z_t dt + \sigma_{1z}(dW^{z,h}+h_z(t)dt) + \sigma_{2z}(dW^{w,h}+h_w(t)dt) + \sigma_{3z}(dW^{f,h}+h_f(t)dt) \tag{A.4}$$

$$Z(UB) = \\ = \{Z\in\mathcal{Z}: \frac{1}{2}h_f^2(t)\le UB_f(t)\}\cap\{Z\in\mathcal{Z}:\frac{1}{2}h_w^2(t)\le UB_w(t)\}\cap\{Z\in\mathcal{Z}:\frac{1}{2}h_f^2(t)\le UB_f(t)\}, \tag{A.5}$$

where ρ is the subjective time discount factor of the investor. The set $\mathcal{Z}$ is a well defined set of probability measures which are absolutely continuous with regard to the benchmark measure Q^0. For an observed three dimensional upper boundary UB, $Z(UB)$ contains all absolutely continuous macroeconomic models that fulfill the three entropy constraints.[13]

We assume that the set of potential models is characterized by

$$UB_z(t) := A_z\eta_z^2(t) \qquad UB_w(t) := A_w\eta_w^2(t) \qquad UB_f(t) := A_f\eta_f^2(t), \tag{A.6}$$

where η_z, η_w, η_f follows a diagonal continuous-time $AR(1)$ process.

The multiple prior consistent recursive version of the robust investor's minimization problem can be solved by the Entropy-constraint Hamilton-Jacobi-Bellman equation. For

[13] Chen and Epstein (2002) contain a detailed analysis of the required conditions.

a log utility investor we have[14]

$$\rho J = \min_{h_f,h_z,h_w} \ln Y_t + \theta_f(t)\left(\frac{1}{2}h_f^2(t) - UB_f(t)\right) + \theta_w(t)\left(\frac{1}{2}h_w^2(t) - UB_w(t)\right) + \\ + \theta_z(t)\left(\frac{1}{2}h_z^2(t) - UB_z(t)\right) + \mathcal{A}^Q J, \tag{A.7}$$

where J is the value function[15],$\theta_f(t) > 0$, $\theta_z(t) > 0$ and $\theta_w(t) > 0$ are the Lagrange multipliers at time t for the three entropy constraints. The expression $\mathcal{A}^Q J$ denotes the second order differential operator applied to the value function J. Economically, this measures the expected continuation utility under the robust macroeconomic model Q. The investor looks for the optimal amount of ambiguity, $h = (h_f, h_w, h_z)$, that minimizes her expected continuation utility subject to the observed entropy constraints.

The solution to the minimization problem coincides with the first-order condition. The derivation of the solution follows Ulrich (2010a) and Ulrich (2010b):

$$h_z(t) = -\sqrt{2 \cdot UB_z(t)}, \quad h_w(t) = \sqrt{2 \cdot UB_w(t)}, \quad h_f(t) = \pm\sqrt{2 \cdot UB_f(t)}. \tag{A.8}$$

We can re-define the constants in the last equation to get:

$$h_f(t) = \pm m_f \eta_f(t), \quad h_z(t) = m_z \eta_z(t), \quad h_w(t) = m_w \eta_w(t), \tag{A.9}$$

where m_f equals $m_f := \sqrt{2A_f} \in R^+$, m_z equals $m_z := -\sqrt{2A_z} \in R^-$ and m_w equals $m_w := \sqrt{2A_w} \in R^+$.

The equilibrium dynamic for the ambiguity premiums follows directly from Ito's

[14]Vissing-Jorgensen (2002) finds that the IES for bondholders is between 0.8 and 1. We therefore set the IES in our bond model to 1. Several other papers restrict $IES = 1$, for example Tallarini (2000), Hansen, Heaton, and Li. (2008), **??**), Piazzesi and Schneider (2006) and Bansal, Kiku, and Yaron (2007). **?**) and Tallarini (2000) do also exploit the analytical tractability of linear value functions.

[15]The value function is particularly convenient because we work with risk aversion of 1. Compare **?**) and Tallarini (2000) for some of the convenient properties. A more general utility function does not substantially change the solution to the minimization problem, because nonlinear value functions are usually linearized to stay tractable (compare Drechsler (2009)).

Lemma:

$$dh_f(t) = \left(m_f a_{\eta_f} + \kappa_{\eta_f} h_f(t)\right) dt + m_f \sigma_{\eta_f} dW^{\eta_f} \tag{A.10}$$

$$dh_z(t) = \left(m_z a_{\eta_z} + \kappa_{\eta_z} h_z(t)\right) dt + m_z \sigma_{\eta_z} dW^{\eta_z} \tag{A.11}$$

$$dh_w(t) = \left(m_w a_{\eta_w} + \kappa_{\eta_w} h_w(t)\right) dt + m_w \sigma_{\eta_w} dW^{\eta_w}. \tag{A.12}$$

A.2 Term Structure of Inflation-Indexed Bonds

The equilibrium price of an inflation-indexed zero-coupon bond $B_t(\tau)$ with time to maturity τ equals the inflation ambiguity adjusted conditional expected value of the intertemporal marginal rate of consumption substitution:

$$B_t(\tau) = e^{-\rho\,\tau} E_t^Q \left[\frac{u_Y(Y_{t+\tau})}{u_Y(Y_t)}\right]. \tag{A.13}$$

Plugging in the log utility function together with the consumption and inflation process and defining $\kappa_t \equiv \rho t + \ln(Y_t)$ yields

$$B_t(\tau) = \frac{1}{\exp(-\kappa_t)} E_t^Q \left[\exp(-\kappa_{t+\tau})\right]. \tag{A.14}$$

The no-arbitrage price at time t of a zero-coupon bond maturing in $t + \tau$ solves the stochastic problem in (A.14). To get a closed-form solution we apply Feynman-Kac's Theorem and solve the dual parabolic PDE:

$$\frac{\partial B(\cdot,\tau)}{\partial \tau} = \mathcal{A} B(\cdot,\tau) \tag{A.15}$$

$$s.t. \quad B(\cdot,0) = 1, \tag{A.16}$$

where $B(\cdot,\tau) \equiv B(\kappa_t, u_t, z_t, h_f(t), h_w(t), h_z(t); \tau)$ and $\mathcal{A}$ represents the second-order differential operator applied to function $B(\cdot,\tau)$. Define $\phi(\kappa_t, u_t, z_t, h_f(t), h_w(t), h_z(t); \tau)$ to be the solution of the stochastic problem:

$$\phi(\kappa_t, u_t, z_t, h_f(t), h_w(t), h_z(t); \tau) = E_t^Q \left[\exp(-\kappa_{t+\tau})\right]. \tag{A.17}$$

Since our economy has logarithmic preferences with an affine consumption process, we guess that the solution has the form:

$$\begin{aligned}&\phi(\kappa_t, u_t, z_t, h_f(t), h_w(t), h_z(t); \tau) = \\ &= e^{-\kappa_t} Z(\tau)\, e^{b_u(\tau)u_t + b_z(\tau)z_t + b_{h_f}(\tau)h_f(t) + b_{h_w}(\tau)h_w(t) + b_{h_z}(\tau)h_z(t)}.\end{aligned} \tag{A.18}$$

If (A.18) solves the stochastic problem than it also solves the PDE

$$\frac{\partial \phi(\cdot, \tau)}{\partial \tau} = \mathcal{A}\phi(\cdot, \tau) \tag{A.19}$$

$$s.t. \quad \lim_{\tau \downarrow 0} \phi(\cdot, \tau) = \exp(-\kappa_t), \tag{A.20}$$

where $\phi(\cdot, \tau) \equiv \phi(\kappa_t, u_t, z_t, h_f(t), h_w(t), h_z(t); \tau)$. Solving the pde gives the result.

A.3 Nominal Term Structure

The equilibrium price of a nominal zero-coupon bond $N_t(\tau)$ with time to maturity τ equals the inflation ambiguity adjusted conditional expected value of the intertemporal marginal rate of consumption substitution times the real payoff at maturity:

$$N_t(\tau) = e^{-\rho\,\tau} E_t^Q \left[\frac{u_Y(Y_{t+\tau})}{u_Y(Y_t)} \frac{p_t}{p_{t+\tau}} \right]. \tag{A.21}$$

Plugging in the log utility function together with the consumption and inflation process and defining $\kappa_t \equiv \rho t + \ln(Y_t)$ yields

$$N_t(\tau) = \frac{1}{\frac{\exp(-\kappa_t)}{p_t}} E_t^Q \left[\frac{\exp(-\kappa_{t+\tau})}{p_{t+\tau}} \right]. \tag{A.22}$$

The no-arbitrage price at time t of a zero-coupon bond maturing in $t + \tau$ solves the stochastic problem in (A.22). To get a closed-form solution we apply Feynman-Kac's Theorem and solve the dual parabolic PDE:

$$\frac{\partial N(\cdot, \tau)}{\partial \tau} = \mathcal{A}N(\cdot, \tau) \tag{A.23}$$

$$s.t. \quad N(\cdot, 0) = 1, \tag{A.24}$$

where $N(\cdot,\tau) \equiv N(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_f(t), h_w(t), h_z(t); \tau)$ and $\mathcal{A}$ represents the second-order differential operator applied to function $N(\cdot,\tau)$. Define $\phi(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_f(t), h_w(t), h_z(t); \tau)$ to be the solution of the stochastic problem:

$$\phi(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_f(t), h_w(t), h_z(t); \tau) = E_t^Q \left[\frac{\exp(-\kappa_{t+\tau})}{p_{t+\tau}} \right]. \tag{A.25}$$

Since our economy has logarithmic preferences with an affine consumption and inflation process, we guess that the solution has the form:

$$\begin{aligned} &\phi(\kappa_t, p_t, u_t, v_t, w_t, h_f(t), h_w(t), h_z(t); \tau) = \\ &= \frac{e^{-\kappa_t} Z(\tau)\, e^{b_u(\tau)u_t + b_v(\tau)v_t + b_w(\tau)w_t + b_z(\tau)z_t + b_{h_f}(\tau)h_f(t) + b_{h_w}(\tau)h_w(t) + b_{h_z}(\tau)h_z(t)}}{p_t}. \end{aligned} \tag{A.26}$$

If (A.26) solves the stochastic problem than it also solves the PDE

$$\frac{\partial \phi(\cdot,\tau)}{\partial \tau} = \mathcal{A}\phi(\cdot,\tau) \tag{A.27}$$

$$s.t. \quad \lim_{\tau \downarrow 0} \phi(\cdot,\tau) = \frac{\exp(-\kappa_t)}{p_t}, \tag{A.28}$$

where $\phi(\cdot,\tau) \equiv \phi(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_f(t), h_w(t), h_z(t); \tau)$. Solving the pde gives the result.

A.4 Detection Error Probability: Not For Publication

The derivation of the detection-error probabilities follows directly from Maenhout (2006). We sketch the main steps in the following.

$$\epsilon_T(m_f, m_w, m_z) = \frac{1}{2}\left(Pr(\ln Q_T > \ln Q_T^0 | Q^0, \mathcal{F}_0) + Pr(\ln Q_T^0 > \ln Q_T | Q, \mathcal{F}_0) \right) \tag{A.29}$$

$$= \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \left(Re\left(\frac{\phi^Q(w,0,T)}{iw} \right) - Re\left(\frac{\phi(w,0,T)}{iw} \right) \right) dw \tag{A.30}$$

where $\phi(.)$ is defined as $\phi(w,0,T) := E\left[e^{i \cdot w \cdot \xi_{1,T}} | \mathcal{F}_0 \right]$ and $\phi^Q(.)$ is defined as $\phi^Q(w,0,T) := E^Q\left[e^{i \cdot w \cdot \xi_{1,T}} | \mathcal{F}_0 \right]$ and $\xi_{1,T} = \ln \frac{dQ_T}{dQ_T^0}$.

Applying Feynman-Kac theorem to ϕ^Q and ϕ reveals that they are an exponentially quadratic function in the amount of ambiguity distortion h_t:

$$\phi^Q(w,t,T) = z_t^{iw+1} e^{G(\tau)+\sum_{i\in\{f,w,z\}} E_i(\tau)h_i(t)+\sum_{i\in\{f,w,z\}} \frac{F_i(\tau)}{2}h_i^2(t)+\sum_{i\neq j;i,j\in\{f,w,z\}} K_{i,j}(\tau)h_i(t)h_j(t)} \tag{A.31}$$

$$\phi(w,t,T) = z_t^{iw} e^{\hat{G}(\tau)+\sum_{i\in\{f,w,z\}} \hat{E}_i(\tau)h_i(t)+\sum_{i\in\{f,w,z\}} \frac{\hat{F}_i(\tau)}{2}h_i^2(t)+\sum_{i\neq j;i,j\in\{f,w,z\}} \hat{K}_{i,j}(\tau)h_i(t)h_j(t)} \tag{A.32}$$

$$z_T := e^{\xi_{1,T}}, \tag{A.33}$$

where $G(\tau), E(\tau), F(\tau), K(\tau), \hat{G}(\tau), \hat{E}(\tau), \hat{F}(\tau), \hat{K}(\tau)$, are deterministic solutions to standard complex valued Riccati equations.

References

Anderson, E., L.P. Hansen, and T.J. Sargent, 2003, A quartet of semigroups for model specification, robustness, prices of risk, and model detection, *Journal of the European Economic Association* 1, 68–123.

Ang, A., G. Bekaert, and M. Wei, 2007, Do macro variables, asset markets or surveys forecast inflation better?, *Journal of Monetary Economics* 54, 1163–1212.

Ang, A., G. Bekaert, and M. Wei, 2008, The term structure of real rates and expected inflation, *Journal of finance* forthcoming.

Ang, A., J. Boivin, S. Dong, and R. Loo-Kung, 2009, Monetary Policy Shifts and the Term Structure, Working Paper.

Ang, A., S. Dong, and M. Piazzesi, 2008, No-Arbitrage Taylor rules, Working Paper.

Bansal, R., D. Kiku, and A. Yaron, 2007, Risks for the long-run: Estimation and inference, Working Paper.

Cagetti, M., L.P. Hansen, T.J. Sargent, and N. Williams, 2002, Robustness and Pricing with Uncertain Growth, *Review of Finanical Studies* 15, 363–404.

Chen, Z., and L. Epstein, 2002, Ambiguity, risk and asset returns in continuous time, *Econometrica* 70, 1403–1443.

Clarida, R., J. Gali, and M. Gertler, 2000, Monetary policy rules and macroeconomic stability: Evidence and some theory, *Quarterly Journal of Economics* 115, 147–180.

Clarida, R., and M. Gertler, 1997, *How the Bundesbank conducts monetary policy.* (in Reducing Inflation: Motivation and Strategy, Romer, C.D. and D.H. Romer, eds., University of Chicago Press, Chicago, 363-406).

Drechsler, Itamar, 2009, Uncertainty, time-varying fear, and asset prices, Working Paper.

Epstein, L., and J. Miao, 2003, A two-person dynamic equilibrium under ambiguity, *Journal of Economic Dynamics and Control* 27, 1253–1288.

Epstein, L., and M. Schneider, 2003, Recursive multiple-priors, *Journal of Economic Theory* 113, 1–31.

Epstein, L., and T. Wang, 1994, Intertemporal asset pricing under Knightian uncertainty, *Econometrica* 62, 283–322.

Ghirardato, P., and M. Marinacci, 2002, Ambiguity made precise: A comparative foundation, *Journal of Economic Theory* 102, 251–289.

Gilboa, I., and D. Schmeidler, 1989, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics* 18, 141–153.

Hansen, L.P., J.C. Heaton, and N. Li., 2008, Consumption strikes back? Measuring long-run risk, *Journal of Political Economy* 116(2), 260–302.

Hansen, L.P., and T.J. Sargent, 2005, Robust estimation and control under commitment, *Journal of Economic Theory* 124, 258–301.

Hansen, L.P., and T.J Sargent, 2007, *Robustness*. (Princeton University Press).

Hansen, L.P., T.J. Sargent, G. Turmuhambetova, and N. Williams, 2005, Robust control and model misspecification, Working Paper.

Klibanoff, P., M. Marinacci, and S. Mukerji, 2005, A smooth modle of decision making, *Econometrica* 73, 1849–1892.

Maenhout, P., 2004, Robust portfolio rules and asset pricing, *Review of Finanical Studies* 17, 951–983.

Maenhout, P., 2006, Robust portfolio rules and detection-error probabilities for a mean-reverting risk premium, *Journal of Economic Theory* 128, N1, 136–163.

Piazzesi, M., and M. Schneider, 2006, Equilibrium yield curves, *2006 NBER Macroeconomics Annual* forthcoming.

Tallarini, T., 2000, Risk sensitive real business cycles, *Journal of Monetary Economics* 45(3), 507–532.

Taylor, J., 1993, Discretion versus policy rules in practice, *Carnegie-Rochester Conference Series on Public Policy* 39, 195–214.

Ulrich, M., 2010a, Inflation ambiguity and the term structure of arbitrage-free U.S. Government bonds, Working Paper.

Ulrich, M., 2010b, Observable long-run ambiguity and long-run risk, Working Paper.

Ulrich, M., 2010c, Uncertaint times ahead, Working Paper.

Vissing-Jorgensen, A., 2002, Limited asset market participation and the elasticity of intertemporal substitution, *Journal of Political Economy* 110, 825–853.

Table 1: PARAMETER ESTIMATES (Standard Errors $\times 1.0e-8$)

Panel A: State Variables

	Drift, κ	Volatility, σ			a
u	-0.024(<0.01)	0.0058(<0.01)			0 (fixed)
v	-0.111(<0.01)	0.0031(<0.01)			0 (fixed)
w	-0.005(<0.01)	0.0040(<0.01)			0 (fixed)
z	-0.081(<0.01)	0.0088 (<0.01)	-0.009(<0.01)	0.0083(<0.01)	0 (fixed)
η_f	-0.256(<0.01)	0.004(<0.01)			0.000718 (<0.01)
η_w	-0.748(<0.01)	0.0042(<0.01)			0.001226 (<0.01)
η_z	-1.870 (0.27)	0.004(<0.01)			0.001293 (<0.01)

Panel B: Growth and Inflation

c_0	0.0065 (fixed)
c_1	1.0 (fixed)
p_0	0.0078 (fixed)
p_1	1.0 (fixed)
σ_{0c}	0.000029 (<0.01)
σ_{1c}	1.98 (<0.01)
σ_{0p}	0.00023 (<0.01)
σ_{1p}	1.58 (0.029)
ρ_{pc}	0.94 (<0.01)
m_f	21.3 (8.5)
m_w	13.6 (0.6)
m_z	-26.42 (48.4)

Table 2: Yield Curve, in %, per quarter

$y^{\$}$

maturity	R	4	8	12	16	20	24	28	32	36	40
data	1.4057	1.4345	1.5096	1.5655	1.6098	1.648	1.6822	1.7115	1.7373	1.7593	1.7789
model	1.4495	1.4345	1.5551	1.5968	1.6277	1.6523	1.6724	1.6888	1.7022	1.7131	1.7789

y^{r}

maturity	20	24	28	32	36
data	0.3782	0.4109	0.4400	0.4650	0.4859
model	0.4175	0.4469	0.4744	0.4996	0.5242

Figure 1: **Nominal Bonds: Ambiguity Premium in** %
The graph presents the cross-sectional estimate of the long-run ambiguity premium and the Taylor rule ambiguity premium.

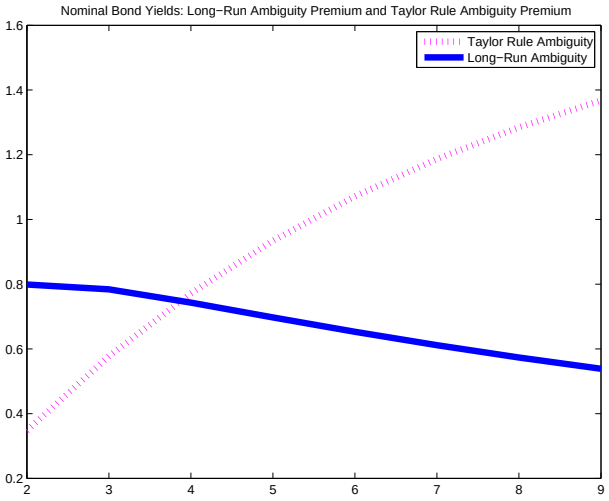

Figure 2: **Inflation-Protected Bonds: Ambiguity Premium in** %
The graph presents the estimated cross-section of the Long-run ambiguity premium and of the Taylor rule ambiguity premium in Tips yields.

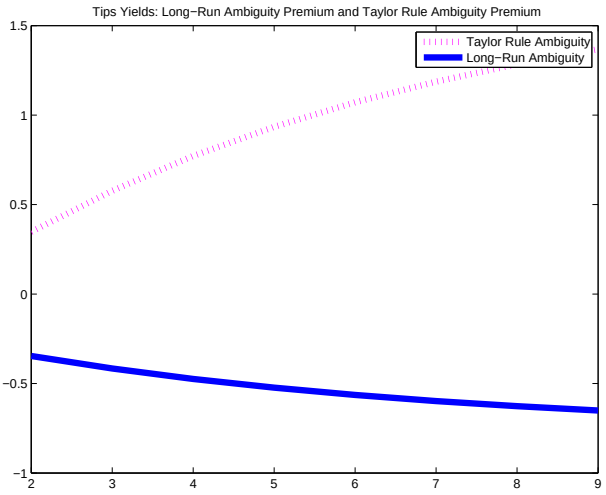

Figure 3: **Components of Nominal Yield Curve , 1981.III - 2009.II**

The left panel presents the nominal yield curve implied cross-section of expected inflation (empirical measure), real yield curve, inflation risk premium and ambiguity premium. The right panel decomposes the ambiguity premium of nominal bond yields into its components: long-run GDP ambiguity premium, long-run inflation ambiguity premium and Taylor Rule ambiguity premium.
Left panel: The solid black line represents the model implied average nominal yield curve. The yellow $.-.$ line shows the average nominal yield curve in the data. The green $-*$ line represents the inflation forecast (empirical measure). The pink $*$ graph represents the estimated Tips yield curve. The solid blue "star" line represents the inflation risk premium. The blue dotted line, ., summarizes the cross-section of the entire ambiguity premium.
Right panel: blue dotted line, ., is the Taylor Rule ambiguity premium; the blue solid line represents the long-run GDP ambiguity premium; the blue $-.$ line represents the long-run inflation ambiguity premium.

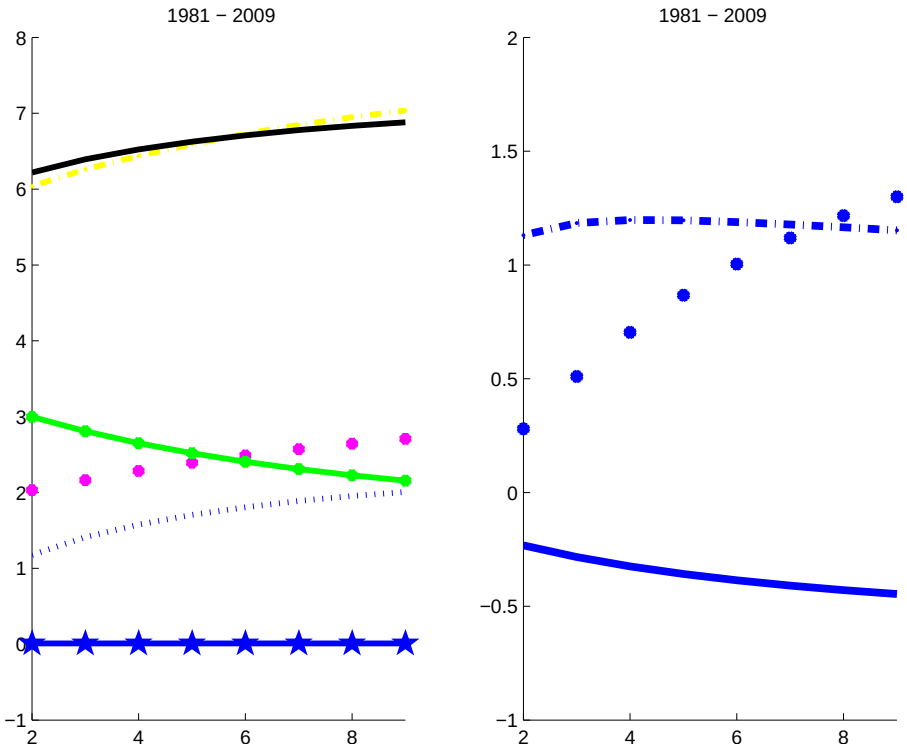

Figure 4: **Components of Real Yield Curve , 1981.III - 2009.II**

The left panel presents the real yield curve without the ambiguity premium (blue dotted curve) and the entire ambiguity premium in Tips yields (red solid line). The right panel decomposes the ambiguity premium of Tips yields into its components: long-run GDP ambiguity premium (blue dotted), long-run inflation ambiguity (blue solid) premium and Taylor Rule ambiguity premium (blue −.).

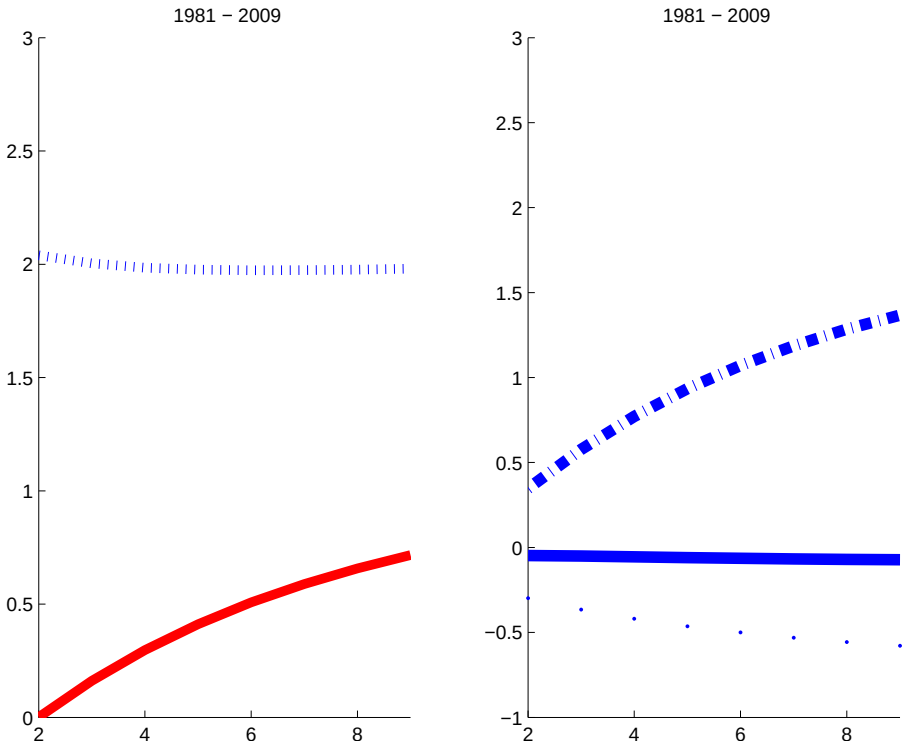

Figure 5: **Empirical vs. Worst-Case Long-Run Inflation Risk, 1981.III - 2009.II**

This figure plots the empirical (solid blue) and robust one-quarter ahead inflation forecast (red $*$) together with the empirical 95% confidence interval (dotted green) for the empirical forecast. The empirical and robust inflation forecast are not distinguishable, neither statistically nor visually.

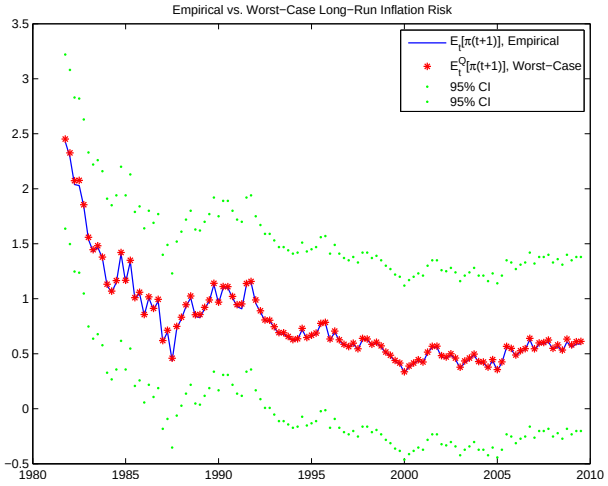

Figure 6: **Empirical vs. Worst-Case Long-Run GDP Risk, 1981.III - 2009.II**

This figure plots the empirical (solid blue) and robust one-quarter ahead GDP growth forecast (red $*$) together with the empirical 95% confidence interval (dotted green) for the empirical forecast. The empirical and robust inflation forecast are not distinguishable, neither statistically nor visually.

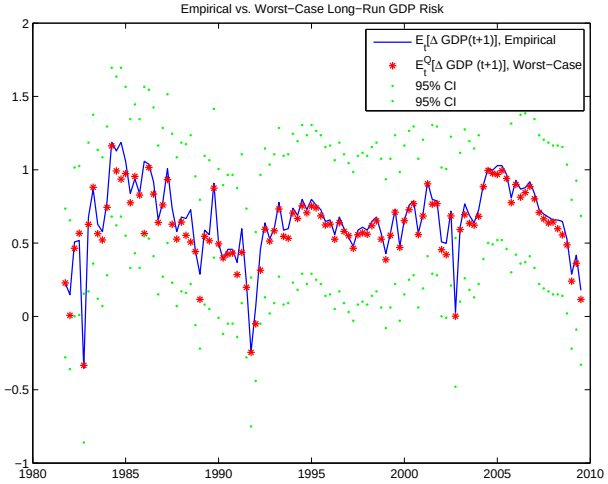

Figure 7: **Five-Year Nominal Yield: Taylor Rule Ambiguity Premium and 95% CI, 1981.III - 2009.II**

These three panels plot the time-series of the ambiguity premium in the five-year nominal yield. The upper panel presents the estimated time-series of the long-run ambiguity premium. That premium is the sum of the long-run inflation ambiguity premium and the long-run GDP ambiguity premium. The middle panel presents the Taylor rule ambiguity premium. The black solid line in the lower panel corresponds to the five-year nominal yield, as observed in the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied five-year nominal yield under the reference model. The reference and worst-case model for the five-year nominal yield are so close to each other, that they are not distinguishable.

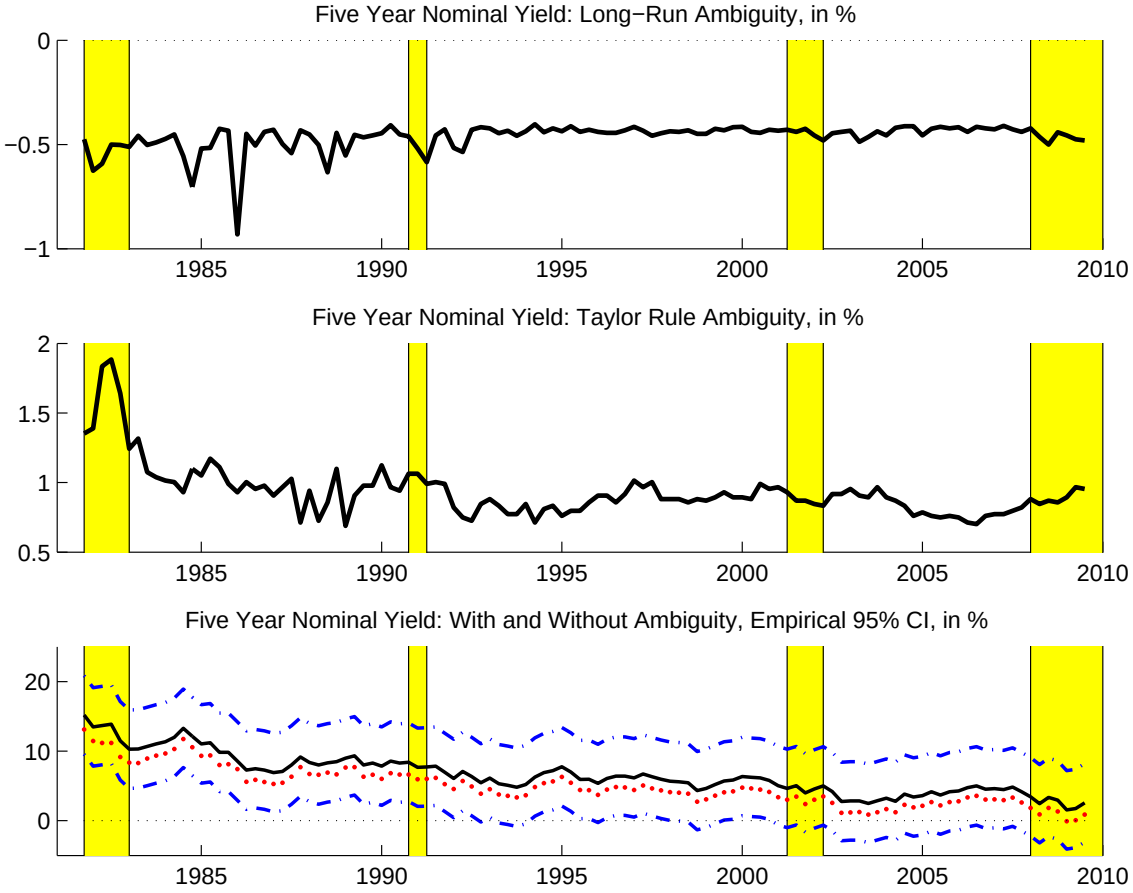

Figure 8: **Ten-Year Nominal Yield: Taylor Rule Ambiguity Premium and 95% CI, 1981.III - 2009.II**

These three panels plot the time-series of the ambiguity premium in the ten-year nominal yield. The upper panel presents the estimated time-series of the long-run ambiguity premium. That premium is the sum of the long-run inflation ambiguity premium and the long-run GDP ambiguity premium. The middle panel presents the Taylor rule ambiguity premium. The black solid line in the lower panel corresponds to the ten-year nominal yield, as observed in the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied ten-year nominal yield under the reference model. The reference and worst-case model for the ten-year nominal yield are very close to each other, making it difficult to distinguish both models

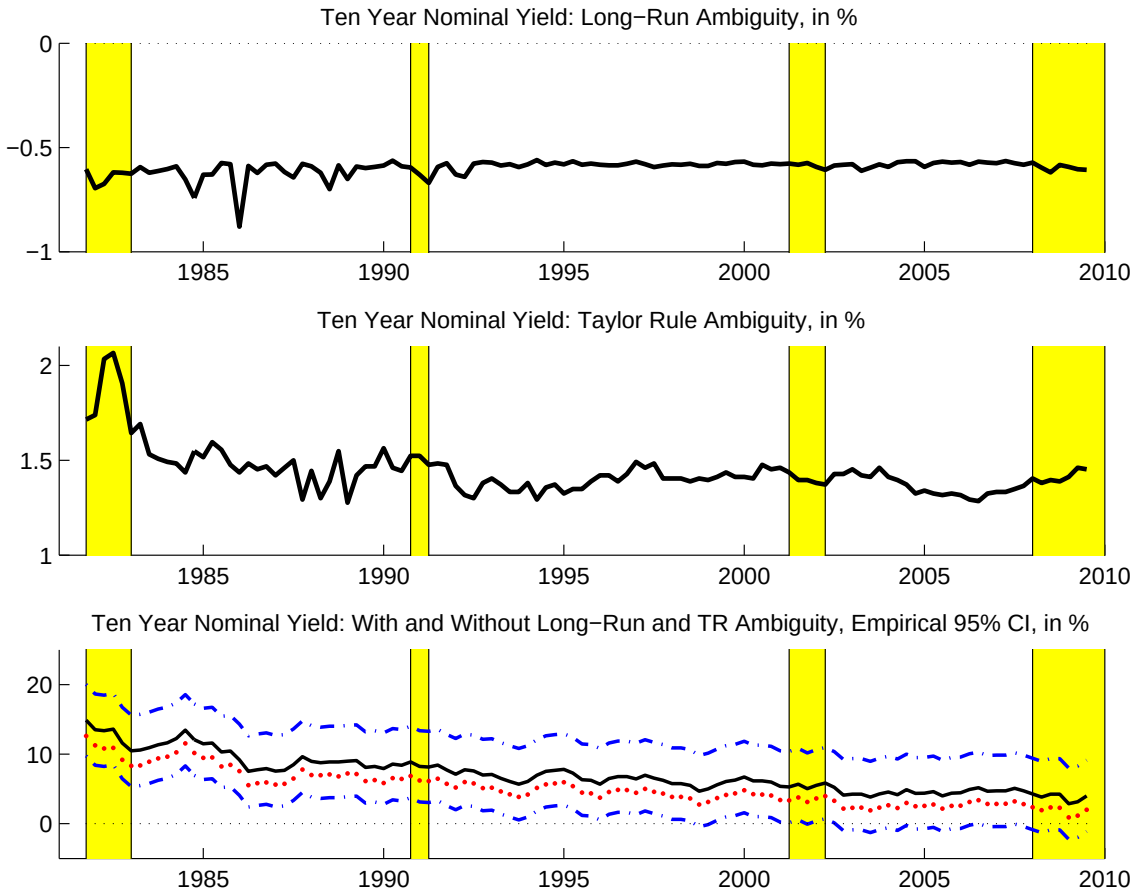

Figure 9: **Five Year Tips Yield: Taylor Rule Ambiguity and 95% CI, 1981.III - 2009.II**

These three panels plot the time-series of the ambiguity premium in the five-year Tips yield. The upper panel presents the estimated time-series of the long-run ambiguity premium. That premium is the sum of the long-run inflation ambiguity premium and the long-run GDP ambiguity premium. The middle panel presents the Taylor rule ambiguity premium. The black solid line in the lower panel corresponds to the five-year Tips yield, as implied by the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied five-year Tips yield under the reference model.

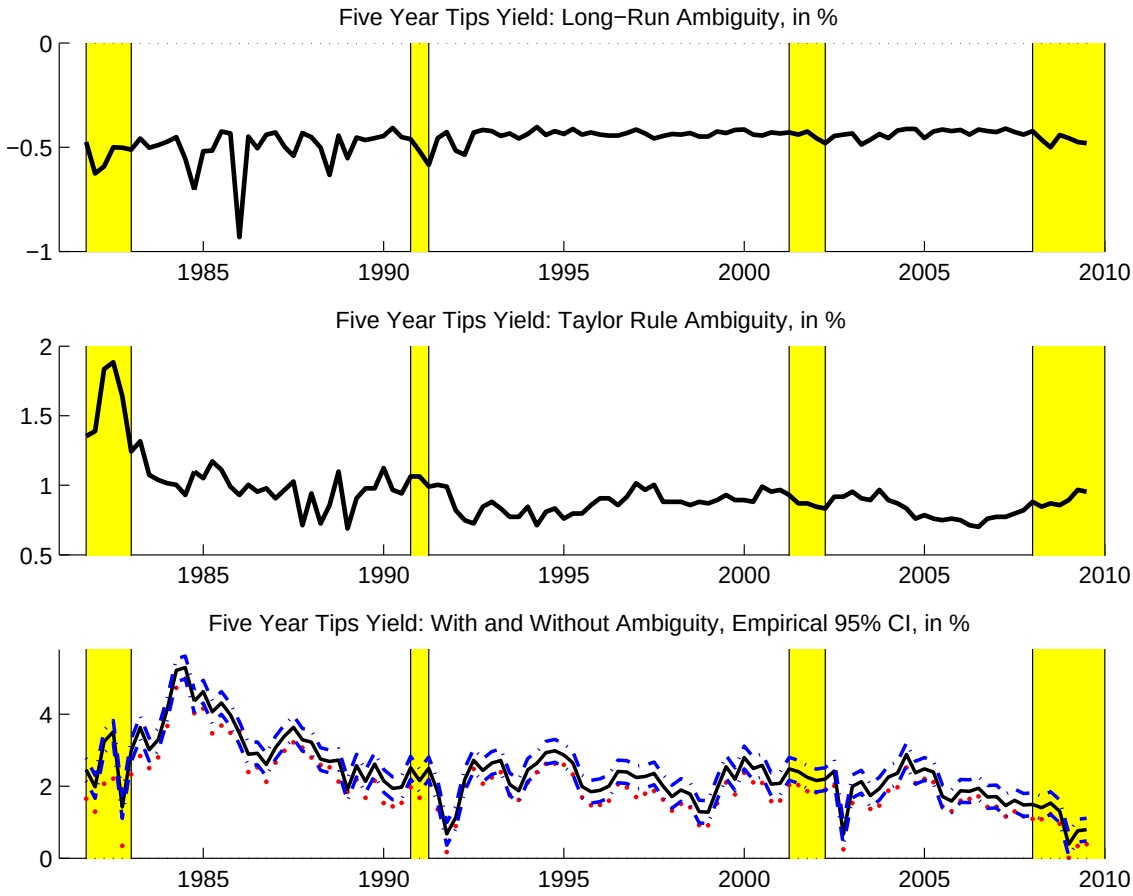

Figure 10: **Model Implied Ten Year Tips Yield with 95% Confidence Interval and Total Ambiguity Premium, 1972.I - 2009.II**

This figure contrasts the model implied ambiguity premium in the ten year Tips yield with the yield itself. The top panel plots the inflation ambiguity premium. The middle panel plots the corresponding GDP ambiguity premium. The lower panel plots the ten year Tips yield, its 95% confidence interval and the ten year Tips yield under the reference measure. The dashed blue lines mark its 95% confidence interval. The latter has been determined based on Tips data from 2003 to 2009. The red dotted line plots the model implied ten year Tips yield under the reference model. The reference and worst-case model for the ten year Tips yield are so close to each other, that they are not distinguishable.