NEAR-OPTIMAL PRICING AND REPLENISHMENT STRATEGIES FOR A RETAIL/DISTRIBUTION SYSTEM

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This paper integrates pricing and replenishment decisions for the following prototypical two-echelon distribution system with deterministic demands. A supplier distributes a single product to multiple retailers, who in turn sell it to consumers. The retailers serve geographically dispersed, heterogeneous markets. The demand in each retail market arrives continuously at a constant rate, which is a general decreasing function of the retail price in the market. The supplier replenishes its inventory through orders (purchases, production runs) from a source with ample capacity. The retailers replenish their inventories from the supplier. We develop efficient algorithms to determine optimal pricing and replenishment strategies for the following three channel structures. The first is the vertically integrated channel, where the system-wide pricing and replenishment strategies are determined by a central planner whose objective is to maximize the system-wide profits. The second structure is that of a vertically integrated channel in which pricing and operational decisions are made sequentially by separate functional departments. The third channel structure is decentralized, i.e., the supplier and the retailers are independent, profit-maximizing firms with the supplier acting as a Stackelberg game leader. We apply our algorithms to a set of numerical examples to quantify the supply chain inefficiencies due to functional segregation or uncoordinated decision making in a decentralized channel. We also gain insight into systematic differences in the associated pricing and operational patterns.

1. INTRODUCTION

Traditional inventory planning models for supply chains assume that all demand processes and hence all revenue streams are exogenously determined. As a consequence, such models focus on the minimization of the logistics costs in the supply chain. Marketing models, on the other hand, focus on pricing strategies and their impact on sales volumes and revenues, typically with a rudimentary and simplistic treatment of the operational costs in the system.

This paper integrates pricing decisions with the determination of replenishment strategies for the following prototypical two-echelon distribution system with deterministic demands. A supplier distributes a single product to multiple retailers who in turn sell it to consumers. The retailers serve geographically dispersed, heterogeneous markets (i.e., the retailers are local monopolists). (A much more complex model arises when the retailers act as oligopolists, where the demand at each retailer is a function of not only his own price but also the prices charged by the other retailers. In a decentralized setting, the problem facing the retailers can be modeled as a noncooperative game, with a solution being based on a particular equilibrium concept. We do not address this type of model, which requires entirely different algorithmic approaches.) The demand in each retail market arrives continuously at a constant rate which is a general decreasing function of the retail price in the market. The supplier replenishes its inventory through orders (purchases, production runs) from a source with ample capacity. The retailers replenish their inventories from the supplier. The logistics costs consist of (1) holding costs for the inventories carried at the supplier and the retailers, (2) fixed and variable costs for orders placed by the supplier and those placed by each of the retailers, and (3) account management costs, i.e., the costs incurred by the supplier for managing each retailer's needs and transactions.

Channel-wide profits are most straightforwardly optimized, if there is a central planner who makes all the pricing and replenishment decisions in the system. A recent paper by Chen et al. (2001) has identified that this centralized solution can in fact be realized in a decentralized system if the channel implements a wholesale pricing scheme that offers discounts based on a number of the retailers' characteristics. However, whether the centralized solution is implemented by a central decision maker specifying all pricing and operational decisions throughout the system, or in a decentralized manner via a properly designed discount scheme, it is necessary to determine the *Centralized solution* efficiently. This represents one of the main objectives of our paper.

Inefficiencies in the above supply chain may arise in a variety of ways. In a vertically integrated channel (i.e., the

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supplier and the retailers belong to the same firm), one often sees that the pricing decisions are made by the marketing department to maximize revenues minus variable production/transportation costs, ignoring less apparent operational (for instance, setup and inventory) costs. Taking these pricing decisions (and the resulting demand rates) as given, the production/distribution department subsequently determines a replenishment policy minimizing total operational costs. In short, the pricing and replenishment decisions are made sequentially. This artificial separation of the marketing and operational decisions, typical of many organizations in which different functional departments act as isolated silos, causes supply chain inefficiencies. We refer to the resulting solution as the *silo solution.*

In a decentralized channel where the supplier and the retailers are independent, profit-maximizing firms, supply chain inefficiencies may be even more severe. Consider the case where the supplier is able to specify the terms (i.e., the wholesale price) unilaterally to maximize its own performance. This situation can be modeled as a Stackelberg game, with the supplier as the leader and the retailers as followers. More specifically, the supplier sets a wholesale price and a replenishment strategy for itself with the objective of maximizing its own profit. Each of the retailers takes the wholesale price as given and determines a retail price and a replenishment strategy so as to maximize its own profits. The resulting solution for the supply chain is referred to as the *Stackelberg solution.*

To understand the magnitudes of the supply chain inefficiencies in the above scenarios as well as any systematic differences in the associated pricing and operational patterns, one must determine the resulting solutions and compare them with the centralized solution. Note that in the silo solution, the channel-wide profits can easily be determined by using existing results (see the literature review below); in fact, this problem mirrors the current academic literature in marketing and operations. However, the problems of obtaining the centralized solution and the Stackelberg solution are considerably harder and to our knowledge, untackled. The main contribution of this paper is to provide efficient algorithms to solve these problems. While not strictly polynomial in N, the number of retailers, we demonstrate that all proposed algorithms are in practice of linear or $O(N \log N)$ time, with instances with 10 retailers solvable in approximately 0.02 CPU seconds on a PC with a Pentium 450-MHz processor.

We apply our algorithms to a set of numerical examples to gain insight into the above mentioned questions and comparisons. Based on the results, we make the following observations. First, the percentage decrease in channel-wide profits when going from the centralized solution to the Stackelberg solution is significant, ranging from around 30% to 100%. When the percentage reaches 100%, it means that the supplier cannot ensure itself of a positive profit, even though a vertically integrated system results in healthy profits. In such cases, the supplier, despite being the price leader, is forced to withdraw from the market. Second, sequential (as opposed to joint) decision making can be quite costly. In some cases, sequential decision making results in negative profits even though the maximum channel profits are positive. Third, compared with the centralized solution, the Stackelberg solution results in higher retail prices and lower sales volumes. Therefore, the double marginalization phenomenon, identified in many industrial organization models since Spengler's (1950) seminal paper, also arises in our Stackelberg game. Finally, we have observed that the centralized solution generally prescribes longer replenishment intervals than the Stackelberg solution. Therefore, shorter replenishment intervals are not automatically the result of the mere integration or coordination of decision making processes, contrary to what some have suggested (Eckstut and Tang 1996).

The remainder of this paper is organized as follows. Section 2 briefly reviews the relevant literature. Section 3 specifies the model and basic notation. The solution methods for the centralized solution, the silo solution, and the Stackelberg solution are developed in §§4, 5, and 6, respectively. Section 7 reports on the numerical examples.

2. LITERATURE REVIEW

It is only slightly more than a decade ago that Roundy (1985) identified effective strategies for the one-warehouse multi-retailer model considered here, under the assumption of centralized control and exogenously given demand rates for all retailers. Even in this simplified setting, the structure of an optimal strategy is exceedingly complex; such fully optimal strategies are therefore of no practical use. Roundy showed, however, that a simple so-called power-of-two policy comes within 2% of being optimal. Under a power-of-two policy, each facility replenishes its inventory with intervals of constant length, which is a power-of-two multiple of a common base-period. Subsequent work (e.g., Roundy 1986, Federgruen et al. 1992, and Federgruen and Zheng 1995) showed that power-of-two policies continue to come within 2% of optimality for a much larger class of production/distribution networks with general cost structures.

The marketing literature on channel coordination focuses on pricing decisions. Jeuland and Shugan (1983) consider a simple channel with one supplier and one retailer. Their model does not consider any inventory replenishment decisions and resulting setup and inventory carrying costs. This single-retailer model is generalized by Ingene and Parry (1995) to multi-retailer settings. The operations literature dealing with channel coordination, on the other hand, has until recently been confined to replenishment decisions, assuming that all demand processes are exogenously given. With the exception of Lal and Staelin (1984), this literature restricts itself to channels with a single retailer (or multiple, identical retailers). These papers compute an optimal centralized solution and discuss various alternatives to implement this solution. Monahan (1984) determines the centralized solution with the restriction that the supplier and retailer use identical order intervals, while Lee and Rosenblatt (1986) relax this assumption. Lal and Staelin (1984)—dealing with multiple, nonidentical retailers—compute a centralized solution implicitly assuming the supplier replenishes its stock infrequently, hence in large quantities. This paper also discusses a heuristic coordination mechanism via an order quantity discount scheme. We refer to Chen et al. (2001) for a more detailed review of this part of the literature.

The need to integrate inventory control and pricing strategies was first advocated by Whitin (1955). Both Whitin (1955) and later Mills (1959, 1962) address a single-period, single-location model in which a single price and supply quantity need to be determined (under uncertain demands). Subsequent work by Karlin and Carr (1962) considers an infinite horizon model, again for a single item and under the assumption that a single constant price is to be specified at the beginning of the planning horizon.

The first treatments of *dynamic* integrated pricing and inventory strategies (i.e., in a multi-period setting) are due to Thomas (1970) and Kunreuther and Schrage (1973), developing variants of the Wagner and Whitin (1958) dynamic lot-sizing model for settings where the demands can be controlled by selecting appropriate price levels. Rajan et al. (1992) analyze a continuous-time version of the same model. Gilbert (1997) extends the results in Kunreuther and Schrage (1973). See Eliashberg and Steinberg (1993) for a comprehensive survey of this area of research. Federgruen and Heching (1999) characterize optimal strategies for a periodic-review model with stochastic demand; see the references therein for earlier treatments of special cases. Finally, there is a stream of recent papers characterizing dynamic pricing strategies in settings with a single initial procurement or perhaps *two* procurements during the planning horizon. These include Bitran and Mondschein (1993, 1997), Bitran et al. (1998), Gallego and van Ryzin (1994), Fisher and Raman (1996), Eppen and Iyer (1997a, 1997b) and Zhao and Zheng (2000). All these models deal with a single item held at a single location only. Most recently, Gilbert (1998) extended the Kunreuther-Schrage model, without setup costs, to allow for multiple items tied together by a joint capacity constraint.

3. MODEL

Consider a distribution system consisting of a supplier who distributes a single product to N geographically dispersed retailers. The consumer demand in each retailer's market occurs continuously at a constant rate, which is determined by the retail price charged in this market, in accordance with a general, time-invariant (i.e., stationary) demand function. All demands must be satisfied without backlogging.

Each firm (i.e., the retailers and the supplier) incurs inventory carrying costs which at any point in time are proportional to its inventory level, at a facility-specific cost rate. The supplier replenishes its inventory through orders from an outside source with unlimited stock, and the retailers place orders with the supplier. All such orders incur fixed and variable order costs; the fixed cost components as well as the variable order cost rates may all be facility-dependent. As with the outside consumer demand, retail orders cannot be backlogged.

In addition, our model allows for a cost component that is often ignored in many inventory models: the supplier may incur a specific annual cost for managing each retailer's needs and transactions. In the consumer electronics industry, for example, one finds that some suppliers establish a "management team" for each of its major retailer accounts, to monitor the retailers' needs, transactions and forecasts, to negotiate and implement sales and logistical terms, etc. Such account teams may comprise logistics managers, and sales and production representatives. The same management team may be devoted exclusively to a single retailer account, for the largest among them, or to multiple accounts, for the smaller ones. For settings where these costs arise, we model the "management costs" associated with a retailer account by a concave function of the retailer's annual sales volume, reflecting economies of scale.

All cost parameters are stationary. Similarly, each retailer maintains a constant retail price. The planning horizon is infinite. For the centralized system, the objective is to optimize long-run average system-wide profits, whereas for the decentralized system, each firms maximizes its own long-run average profits.

Define for $i = 1, \ldots, N$,

p_i = retail price charged by retailer i
$d_i(p_i)$ = annual consumer demand in the market served by retailer i, a strictly decreasing function of p_i
K_0 = fixed (setup) cost per order placed by the supplier
K_i = fixed (setup) cost per order placed by retailer i
h_0 = basic annual holding cost per unit carried in inventory anywhere in the system
h_i = incremental (or echelon) annual holding cost per unit carried in inventory at retailer i.
$\bar{h}_i = h_0 + h_i$
c_0 = variable cost per unit ordered by the supplier
c_i = variable cost per unit ordered by retailer i
$\psi(d_i)$ = annual cost for managing retailer i's account, with $\Psi(\cdot)$ increasing, concave and $\Psi(0) = 0$

(In this paper, we use "increasing" and "decreasing" in the weak sense.) Let $p_i(d_i)$ be the inverse demand function, since the demand function $d_i(\cdot)$ is strictly decreasing. We assume that $p_i(d_i)d_i$ is strictly concave and differentiable in d_i and that the price p_i is allowed to vary over a finite interval $[p_i^{\min}, p_1^{\max}]$ only.

Note that the quantities $\{K_i, h_i, c_i; i = 0, \ldots, N\}$ are given parameters while the prices $\{p_i; i = 1, \ldots, N\}$ or alternatively the demand rates $\{d_i; i = 1, \ldots, N\}$ are decision variables. We assume, as in standard inventory models, that for all $i = 1, \ldots, N$, $h_i \geqslant 0$, i.e., the cost of carrying a unit at retailer i is at least as large as the cost of carrying it in the supplier's warehouse. In addition, the above specified costs for retailers' orders, as represented by the parameters $\{K_i, c_i; i = 1, \ldots, N\}$, reflect the total expenditure associated with retailer orders for the system as a whole, whether expended by the supplier, the retailer or both. Note that these costs include all logistics costs such as handling and transportation costs, but exclude any transfer payments between the retailers and the supplier. (For each order by retailer i, it is often the case that the supplier incurs a fixed order-processing cost K_i^s and the retailer incurs a fixed setup cost K_i^r with $K_i^s + K_i^r = K_i$. The separation of these cost components of K_i is clearly unnecessary in analyzing the centralized system, and it does not fundamentally affect the analysis of the decentralized system, see below.)

We assume, for convenience, that all orders are received instantaneously upon placement. Positive, deterministic leadtimes can be handled by a simple shift in time of all desired replenishment epochs, as determined under zero leadtimes, by the actual leadtimes.

Assume, for a moment, that all retailer prices and hence all demand rates and revenues are given, and consider the problem of finding an integrated replenishment strategy minimizing system-wide long-run average setup and holding costs. As mentioned in the introduction, Roundy (1985) showed that even for this limited problem an exact optimal strategy is exceedingly complex and entirely intractable. On the other hand, he showed that a simple power-of-two policy is guaranteed to come within 2% of optimality and that the parameters of this policy can be computed by a simple algorithm in $O(N \log N)$ time. (A later refinement by Queyranne 1987 reduces the complexity to $O(N)$.) Under such a power-of-two policy a base period T_L is chosen, and each of the facilities chooses a constant replenishment interval from the set of the power-of-two multiples of T_L, receiving replenishments when its inventory is down to zero. Let

T_i = replenishment interval for retailer i, $i = 1, \ldots, N$

T_0 = replenishment interval for the supplier.

Throughout this paper, we assume that T_L is fixed. A power-of-two policy is one where

$$T_i = 2^{m_i} T_L, \quad i = 0, \ldots, N, \tag{1}$$

with m_i a (positive or negative) integer. (The above 2% optimality gap is obtained when T_L is allowed to vary. If T_L is fixed, the guaranteed optimality gap increases to 6%.)

Assume that the system starts out empty (i.e., without inventories). Thus all facilities place an order at time 0. Note that thereafter all retailers with a given replenishment interval $T_i = t$ place all of their orders at the same replenishment epochs along with all retailers with a *smaller* replenishment interval. (A replenishment interval t' smaller than t satisfies $t = 2^m t'$, due to Equation (1), with m a positive integer.) The power-of-two structure thus induces maximum coordination between the replenishment orders of different facilities, providing an intuitive explanation for its excellent performance.

4. THE CENTRALIZED SYSTEM

This section considers the problem facing a central planner for the supply chain, who determines the retail prices and the system-wide replenishment strategy, and whose objective is to maximize the system-wide profits. We develop an efficient algorithm for solving this problem.

The problem considered here is considerably harder than that treated in Roundy (1985), who showed that for exogenously given demand rates, there exists a power-of-two policy that comes within 2% of optimality (or 6% if the base period is fixed). This negligible optimality gap, when measured in terms of cost differentials along with other attractive features discussed above and elsewhere (e.g., Roundy 1985 and Federgruen et al. 1992), motivates us to restrict ourselves to replenishment strategies which are of the power-of-two type. Below, we examine the optimality gap (in terms of system-wide profits) resulting from this restriction. Let profit* = revenue* − cost* be the maximum system-wide profits under an optimal pricing/replenishment strategy without the power-of-two restriction, and profit^0 the maximum system-wide profits under the restriction. (Here the revenue term is defined to be sales minus variable costs minus account management costs, and the cost term refers to setup and inventory holding costs only. Note that the supply chain's revenue is completely determined by the demand rates.) Now fix the demand rates (thus retail prices) at those found in the (truly) optimal pricing/replenishment solution, and consider the remaining problem of choosing a replenishment strategy to minimize the system-wide inventory costs (i.e., setup and inventory holding costs). This problem is exactly the one considered in Roundy (1985). Let LB be a lower bound on the system-wide inventory costs of any feasible policy. As mentioned, one can find an optimal power-of-two policy so that the corresponding system-wide inventory costs, denoted by cost$'$, is within 6% of LB (assuming the base period is fixed), i.e., cost$' \leqslant 1.06LB$. Because cost$^* \geqslant LB$, 1.06cost$^* \geqslant$ cost$'$. Let profit$'$ be the resulting system-wide profits. Because the demand rates are fixed at those found in the truly optimal solution, the total revenue for this system is still revenue*. Thus profit$'$ = revenue* − cost$'$. Note that this sequential solution (fix demand rates first, then optimize within the class of power-of-two policies) is only a feasible solution to the integrated pricing/replenishment problem under the power-of-two restriction. Thus profit$' \leqslant$ profit^0. The above observations lead to the following inequalities:

$$\frac{\text{profit}^0}{\text{profit}^*} \geqslant \frac{\text{revenue}^* - \text{cost}'}{\text{profit}^*} = 1 - \frac{\text{cost}' - \text{cost}^*}{\text{profit}^*}$$
$$\geqslant 1 - 0.06\frac{\text{cost}^*}{\text{profit}^*}.$$

Therefore, if the supply chain has a relatively large profit margin (corresponding to a relatively large value of profit*/cost*), we can expect the profit gap caused by the power-of-two restriction to be quite small. Conversely, if the profit margin is small, the profit gap can be large. Nevertheless, power-of-two policies represent the state-of-the-art solution for the kind of replenishment problem considered here.

Recall from Roundy (1985) that under a power-of-two policy the long-run average (system-wide) setup and holding costs are given by

$$\frac{K_0}{T_0}+\sum_{i=1}^{N}\left\{\frac{K_i}{T_i}+\frac{1}{2}h_0d_i(p_i)\max\{T_0,T_i\}+\frac{1}{2}h_id_i(p_i)T_i\right\}. \quad (2)$$

Treating $\{d_i, i=1,\ldots,N\}$ as the decision variables, along with the intervals $\{T_i, i=0,\ldots,N\}$, we obtain the following expression for the system-wide profits Π:

$$\Pi=-\frac{K_0}{T_0}+\sum_{i=1}^{N}G_i(d_i,T_i,T_o), \quad (3)$$

where

$$G_i(d_i,T_i,T_0)=(p_i(d_i)-c_0-c_i)d_i-\Psi(d_i)-\frac{K_i}{T_i}-\frac{1}{2}h_0d_i\max\{T_0,T_i\}-\frac{1}{2}h_id_iT_i.$$

The problem thus reduces to determining a power-of-two vector T satisfying Equation (1) and a vector of demand rates d achieving the maximum in Equation (3). Let $(T_0^*, T_1^*, \ldots, T_N^*)$ and $(d_1^*, \ldots, d_N^*)$ be the optimal solution, and Π^* the maximum channel profits.

Below, we develop an efficient algorithm to compute an optimal vector pair (T, d) maximizing channel-wide profits. We first need the following preliminary results. For any fixed vector of demand rates d, define for $i=0,\ldots,N$

$\widehat{T}_i(d)=$ the replenishment interval for facility i which achieves the unconstrained minimum of Equation (2), i.e., without consideration of the constraints in Equation (1). (Roundy 1985 establishes the existence of a unique minimum of Equation (2) for any given d.)

$T_i^*(d)=$ the smallest replenishment interval for facility i which achieves the minimum of Equation (2) subject to the power-of-two constraints (1).

It follows from Roundy (1985) that $T_i^*(d)$ is obtained from $\widehat{T}_i(d)$ by rounding the latter to the power-of-two multiple of T_L that is closest in the *relative* sense. More formally, let $m=\lfloor\log_2(\widehat{T}_i(d)/T_L)\rfloor$, where $\lfloor y\rfloor$ is the largest integer less than or equal to y. Then, by the definition of m, $2^mT_L\leqslant \widehat{T}_i(d)<2^{m+1}T_L$, and

$$T_i^*(d)=\begin{cases}2^mT_L & \text{if } \widehat{T}_i(d)\leqslant\sqrt{2}2^mT_L,\\ 2^{m+1}T_L & \text{otherwise.}\end{cases} \quad (4)$$

In the sequel, "rounding" refers to the above procedure.

Similarly, for any fixed vector of power-of-two intervals $T=(T_0,T_1,\ldots,T_N)$, let $d_i^*(T)=$ the demand rate for retailer i which achieves the maximum of Equation (3), $i=1,\ldots,N$.

The following lemma identifies some useful monotonicity properties. We say a function $f(x)$ is decreasing in the N-dimensional vector x if $f(x')\geqslant f(x'')$ for all x' and x'' with $x_i'\leqslant x_i''$ for $i=1,\ldots,N$.

LEMMA 1. (a) *$T_i^*(d)$ is decreasing in d for $i=0,\ldots,N$.*
(b) *$d_i^*(T)$ is decreasing in T for $i=1,\ldots,N$.*

PROOF. (a) Observe first that the rounding procedure transforming $\widehat{T}$ to T^* is order-preserving, i.e., if $\widehat{T}_i(d^1)\leqslant\widehat{T}_i(d^2)$ for some demand rate vectors d^1 and d^2 and some $i=0,\ldots,N$ then $T_i^*(d^1)\leqslant T_i^*(d^2)$. It thus suffices to prove that $\widehat{T}(d)$ is decreasing in d.

We first derive an equivalent formulation for Equation (2). Introduce auxiliary variables $T_{0i}=\max\{T_0,T_i\}$, $i=1,\ldots,N$. The problem of minimizing Equation (2) is thus equivalent to the following convex program:

$$(P)\min_{T_i,T_{0i}}\quad \sum_{i=0}^{N}\frac{K_i}{T_i}+\sum_{i=1}^{N}\frac{1}{2}h_0d_iT_{0i}+\sum_{i=1}^{N}\frac{1}{2}h_id_iT_i, \quad (5)$$

$$\text{s.t.}\quad T_{0i}\geqslant T_0,\quad i=1,\ldots,N \quad (6)$$

$$T_{0i}\geqslant T_i,\quad i=1,\ldots,N. \quad (7)$$

(Because for any $i=1,\ldots,N$ the objective (5) is increasing in T_{0i}, in an optimal solution to (P) at least one of the two constraints is binding, i.e., $T_{0i}=\max\{T_o,T_i\}$.) Next, substitute the T_i and T_{0i} variables by $x_i=T_i^{-1}$ and $x_{0i}=T_{0i}^{-1}$:

$$(P)\min_{x_i,x_{0i}}\quad f(x,d)\stackrel{\text{def}}{=}\sum_{i=0}^{N}K_ix_i+\sum_{i=1}^{N}\frac{1}{2}h_0d_i/x_{0i}+\sum_{i=1}^{N}\frac{1}{2}h_id_i/x_i, \quad (8)$$

$$\text{s.t.}\quad x_{0i}\leqslant x_0,\quad i=1,\ldots,N \quad (9)$$

$$x_{0i}\leqslant x_i,\quad i=1,\ldots,N. \quad (10)$$

Let:

$$S=\{x=(x_0,x_1,\ldots,x_N;x_{01},\ldots,x_{0N})\in R^{2N+1}:x\geqslant 0 \text{ satisfies (9) and (10)}\},$$

$$D=\{d\in R^N:d\geqslant 0\},$$

$x_d^*=$ the vector $x\in S$ which achieves the minimum in Equation (8). (Because $\widehat{T}(d)$ is the unique minimizer of (2) for fixed d, x_d^* is uniquely determined as well.)

Note that D is a partially ordered set. Also, S is a lattice, since it is a partially ordered set and it can be shown that if $x,y\in S$ then $\min\{x,y\}\in S$ and $\max\{x,y\}\in S$. ($\min\{x,y\}$ is the vector of component-wise minima and

$\max\{x, y\}$ the vector of component-wise maxima.) We verify that $\min\{x, y\} \in S$. (The proof that $\max\{x, y\} \in S$ is analogous.) Because $x \in S$, for all $i = 1, \ldots, N$, $x_0 \geqslant x_{0i} \geqslant \min\{x_{0i}, y_{0i}\}$ and $y \in S$, $y_0 \geqslant y_{0i} \geqslant \min\{x_{0i}, y_{0i}\}$. Thus, $\min\{x_0, y_0\} \geqslant \min\{x_{0i}, y_{0i}\}$, verifying Equation (9). The proof that $\min\{x, y\}$ satisfies Equation (10) as well is analogous.

The fact that x_d^* is increasing in $d \in D$ and hence Lemma 1 follows from Theorem 6 in Topkis (1978) by verifying that:

(i) $f(x, d)$ is submodular in $x \in S$ for each $d \in D$

(ii) $f(x, d)$ has antitone differences in (x, d) on $S \times D$.

Part (i) is immediate because $f(x, d)$ is in fact a modular, i.e., a separable function in the x-variables. To verify (ii), we need to show that for all $x^1 \leqslant x^2$ and $d^1 \leqslant d^2$:

$$f(x^2, d^2) - f(x^1, d^2) \leqslant f(x^2, d^1) - f(x^1, d^1).$$

Note that the corresponding inequality holds term by term in Equation (8).

(b) Note first that for fixed T, the profit function in Equation (3) is separable in $d_i, i = 1, \ldots, N$. More specifically, d_i is chosen to maximize

$$(p_i(d_i) - c_0 - c_i)d_i - \Psi(d_i) - \left(\frac{1}{2}h_0 \max\{T_0, T_i\} + \frac{1}{2}h_i T_i\right)d_i,$$

which can be written as

$$r_i(d_i) - \tilde{c}_i d_i, \tag{11}$$

where $r_i(d_i) = (p_i(d_i) - c_0 - c_i)d_i - \Psi(d_i)$ and $\tilde{c}_i = \frac{1}{2}h_0 \max\{T_0, T_i\} + \frac{1}{2}h_i T_i$, which is clearly increasing in T. Therefore, it suffices to show that the value of d_i that maximizes the value of Equation (11) is decreasing in $\tilde{c}_i$. Take any $\tilde{c}_i^1 < \tilde{c}_i^2$. Let d_i^1 and d_i^2 be the corresponding optimal values of d_i. Note that for any $d > d_i^1$, we have

$$r_i(d_i^1) - \tilde{c}_i^1 d_i^1 \geqslant r_i(d) - \tilde{c}_i^1 d > r_i(d) - \left[\tilde{c}_i^1 d_i^1 + (d - d_i^1)c_i^{-2}\right],$$

which is equivalent to

$$r_i(d_i^1) - \tilde{c}_i^2 d_i^1 > r_i(d) - \tilde{c}_i^2 d.$$

Thus $d_i^2 \leqslant d_i^1$. $\square$

The above result is quite intuitive. We can interpret $\tilde{c}_i$ as the "imputed" marginal cost for retailer i. As the marginal cost increases, the retailer tends to charge a higher price and sell less.

We now derive bounds on $\{T_i^*(d), i = 0, \ldots, N\}$ that apply to all achievable demand vectors d. Let $d_i^{\max} = d_i(p_i^{\min})$ and $d_i^{\min} = d_i(p_i^{\max})$ for $i = 1, \ldots, N$. Define

$$\tau_i' = \sqrt{\frac{2K_i}{\bar{h}_i d_i^{\max}}} \quad \text{and} \quad \tau_i = \sqrt{\frac{2K_i}{h_i d_i^{\min}}}, \quad i = 1, \ldots, N,$$

and

$$\tau_0' = \min\{\tau_i', i = 1, \ldots, N\} \quad \text{and}$$

$$\tau_0 = \max\{\tau_i, i = 1, \ldots, N; z\},$$

where $z = \sqrt{2K_0/(h_0 \sum_{i=1}^N d_i^{\min})}$. For $i = 0, 1, \ldots, N$, let $\bar{\tau}_i'$ and $\bar{\tau}_i$ be the rounded values of τ_i' and τ_i, respectively.

Lemma 2. $\bar{\tau}_i' \leqslant T_i^*(d) \leqslant \bar{\tau}_i, i = 0, 1, \ldots, N$.

Proof. Fix the demand vector d. Thus the revenue is fixed, and profit maximization reduces to minimizing the cost function in Equation (2). Take any $i = 1, \ldots, N$. Because

$$\sqrt{\frac{2K_i}{\bar{h}_i d_i}} \leqslant \widehat{T}_i(d) \leqslant \sqrt{\frac{2K_i}{h_i d_i}}$$

(see, e.g., Roundy 1985) and $d_i^{\min} \leqslant d_i \leqslant d_i^{\max}$, we have $\tau_i' \leqslant \widehat{T}_i(d) \leqslant \tau_i$. The lemma follows because rounding (to a power-of-two number) is order-preserving. We next prove the lemma for $i = 0$. Because $\widehat{T}_0(d) \geqslant \min_{i=1,\ldots,N} \widehat{T}_i(d) \geqslant \tau_0'$ (see, e.g., Roundy 1985 for the first inequality), the lower bound on $T_0^*(d)$ follows. To prove the upper bound, suppose $T_i = T_i^*(d)$ for $i = 1, \ldots, N$. For $T_0 \geqslant \max\{\bar{\tau}_i, i = 1, \ldots, N\}$ which implies $T_0 \geqslant T_i$ for $i = 1, \ldots, N$, the only part in Equation (2) that depends on T_0 is

$$\frac{K_0}{T_0} + \frac{1}{2}h_0 \sum_{i=1}^N d_i T_0, \tag{12}$$

which is convex and its unconstrained minimum is $x \stackrel{\text{def}}{=} \sqrt{2K_0/(h_0 \sum_{i=1}^N d_i)} \leqslant z$. If $z \leqslant \max\{\bar{\tau}_i, i = 1, \ldots, N\}$ then Equation (12) is increasing in T_0 over the range $T_0 \geqslant \max\{\bar{\tau}_i, i = 1, \ldots, N\}$, in which case $T_0^*(d) \leqslant \max\{\bar{\tau}_i, i = 1, \ldots, N\}$; otherwise, $T_0^*(d)$ does not exceed the rounded value of z. The lemma follows by combining these two cases. $\square$

We are now ready to present an algorithm for solving the joint optimization problem. Recall from Equation (3) that for a given T_0, the channel-wide profit function is separable in (d_i, T_i) for $i = 1, \ldots, N$. The algorithm therefore has two loops. The first loop iterates over all values of T_0 within the bounds identified in Lemma 2. Within this loop, for each retailer i and each value of T_i in $[\bar{\tau}_i', \bar{\tau}_i]$, we optimize the profit function $G_i(d_i, T_i, T_0)$ over d_i. Let n_i be the number of power-of-two values between $\bar{\tau}_i'$ and $\bar{\tau}_i, i = 0, 1, \ldots, N$. Initially, the optimization over d_i is over the range $[d_i^{\min}, d_i^{\max}]$. For each retailer i, we keep track of n_i upper bounds on d_i, $\{\bar{d}_{ij} : j = 1, \ldots, n_i\}$, where $\bar{d}_{ij}$ applies when T_i is set at its jth smallest value. By Lemma 1's monotonicity property, we have $\bar{d}_{ij} \geqslant \bar{d}_{i,j+1}$ for all $j = 1, \ldots, n_i - 1$. By the same monotonicity property, any upper bound $\bar{d}_{ij}$ which applies for a given value of T_0, continues to apply when T_0 is replaced by a larger value, in particular as T_0 is replaced by $2T_0$ in the outer loop. As a result, when executing the $(j+1)$st iteration of the inner loop for retailer i, we set $\bar{d}_{i,j+1}$ equal to $\bar{d}_{ij}$ or the current value of $\bar{d}_{i,j+1}$, whichever is smaller.

ALGORITHM I.

Initialize: $\Pi^* := 0$;
For $i := 1, \ldots, N$ do $\underline{d}_i := d_i^{\min}$;
For $i := 1, \ldots, N$ and $j := 1, \ldots, n_i$ do
$\bar{d}_{ij} := d_i^{\max}$;
$T_0 := \bar{\tau}_0'$;
Step 0: $\Pi := -K_0/T_0$;
Step 1: For $i := 1, \ldots, N$ do
begin
$j := 1$; $T_i := \bar{\tau}_i'$; $\Pi_i := -\infty$;
Iterative Step:
Determine the smallest maximum x of
$G_i(x, T_i, T_0)$ on the interval $[\underline{d}_i, \bar{d}_{ij}]$;
$\bar{d}_{ij} := x$;
If $G_i(x, T_i, T_0) > \Pi_i$ then
begin
$T_i^0 = T_i$;
$d_i^0 := x$;
$\Pi_i := G_i(x, T_i, T_0)$;
end;
If $j < n_i$ then
begin
$\bar{d}_{i,j+1} := \min\{\bar{d}_{i,j+1}, \bar{d}_{ij}\}$;
$j := j + 1$; $T_i := 2T_i$; and go to
Iterative Step;
end;
$\Pi := \Pi + \Pi_i$
end;
If $\Pi > \Pi^*$ then
begin
$\Pi^* := \Pi$;
$T_0^* := T_0$;
For $i := 1, \ldots, N$ do $T_i^* := T_i^0$ and
$d_i^* := d_i^0$;
end;
$T_0 := 2T_0$;
If $T_0 \leqslant \bar{\tau}_0$ then go to *Step 0*.

The computational effort of Algorithm I consists primarily of the multiple optimizations of the single-variable function $G_i(x, T_i, T_0)$ over x. These functions are often strictly concave, e.g., in the absence of account management costs ($\Psi(\cdot) = 0$) or when the function $\Psi(\cdot)$ is of the fixed-plus-linear type. In this case, maximizing G_i is entirely straightforward, e.g., by bisection or by finding the unique root of its derivative, while a more general single-variable optimization technique is required in case G_i fails to be concave. As is the case with all optimization problems involving general, nonlinear functions, we refer to the single-variable maximization procedure optimizing G_i as an "oracle" and we assess the complexity of the algorithm in terms of the number of oracle calls required. Note that the outer loop in Algorithm I is traversed n_0 times and the inner loop $\sum_{i=1}^N n_i$ times. Each execution of the inner loop involves a single call to the oracle and a constant number of multiplications and comparisons. The overall effort is thus given by $n_0 \sum_{i=1}^N n_i$ oracle calls and $O(n_0 \sum_{i=1}^N n_i)$ elementary operations. The expression $n_o \sum_{i=1}^N n_i$ fails to be polynomial for completely arbitrary parameters. However, as has been frequently observed for other algorithms involving power-of-two policies (see, e.g., Federgruen and Zheng 1995), it would be exceedingly rare to have an instance where more than (say) eight distinct power-of-two values need to be investigated for any of the facilities, because otherwise, the largest considered replenishment interval would be more than 500 times the smallest considered interval.Thus, for all practical purposes, the complexity of Algorithm I can be conservatively bounded by $64N$ calls to the oracle and $O(N)$ elementary operations. This bound does not incorporate the very significant shortcuts achieved by the updating of the upper bounds $\{\bar{d}_{ij}\}$ described above.

5. THE SILO SOLUTION

The silo solution assumes that the supplier and the retailers are part of a single firm, where pricing and replenishment decisions are delegated to separate departments. More specifically, the marketing department first determines the demand rates (or the retail prices) to maximize the firm's gross profits $\sum_{i=1}^N (p_i(d_i) - c_0 - c_i)d_i$. Let d_i^s be the optimal demand rate for retailer i, $i = 1, \ldots, N$. Given these demand rates, the production/distribution department then determines the replenishment intervals to minimize the firm's logistics costs. This problem is identical to the problem Roundy (1985) has solved; thus one can use his $O(N \log N)$ algorithm to identify an optimal power-of-two policy. (An alternative implementation by Queyranne 1987 results in a linear time algorithm.) Alternatively, one can use a modified version of Algorithm I. First, redefine

$$\tau_i' = \sqrt{\frac{2K_i}{\bar{h}_i d_i^s}} \quad \text{and} \quad \tau_i = \sqrt{\frac{2K_i}{h_i d_i^s}}, \quad i = 1, \ldots, N,$$

and $z = \sqrt{2K_0/(h_0 \sum_{i=1}^N d_i^s)}$. The rounded values of τ_i' and τ_i are again the lower and upper bounds on $T_i^*(d^s)$, $i = 0, 1, \ldots, N$, where $d^s = (d_1^s, \ldots, d_N^s)$. With these updated bounds, one can use Algorithm I to find the silo solution by skipping the step optimizing over the demand rates. Therefore, this algorithm searches over all possible values of the replenishment intervals. Because no oracle calls are required for the silo solution, the complexity of this modified version of Algorithm I is $O(n_0 \sum_{i=1}^N n_i)$. As argued above, in most practical instances $n_i \leqslant 8$ for $i = 0, 1, \ldots, N$, resulting in a linear time algorithm.

6. THE STACKELBERG GAME

In this section, we assume that the supplier and the retailers are independent firms, each of which maximizes its own profits. The supplier, in anticipation of the retailers' reactions, chooses a constant wholesale price and a replenishment strategy for itself with the objective of maximizing its own profits. The retailers take the wholesale price as given and each of them maximizes its individual profits by charging an optimal retail price and following an optimal replenishment strategy. In other words, the players in the distribution channel play a Stackelberg game, with the supplier as the leader, and the retailers as followers. This *modus operandi* represents many traditional distribution channels in which the channel members fail to coordinate their decisions. Nevertheless, no solution procedure has been proposed to solve this decentralized system. The objective of this section is to develop an efficient algorithm for the Stackelberg game. In addition to providing the optimal policy parameters for such decentralized systems, the algorithm enables us to quantify the relative benefits of the centralized solution.

Every decentralized supply chain requires an upfront specification of a *contract*, i.e., a set of ground rules for the commercial interactions between the different parties involved. Such a contract may involve the specification of a pricing rule, the commitment to deliver in whole or in part (possibly within a specified leadtime), return policies, restrictions on the times at which orders may be placed and delivered (e.g., daily, every Tuesday or Friday, etc.), among others. In our model, the contract consists of the following provisions. We assume, without loss of generality, that the system is without inventory at time 0. The supplier commits himself to satisfy all retailer orders in their entirety without backlogging and to deliver them with a fixed leadtime, which is without loss of generality normalized to be zero. The supplier is responsible for all (fixed and variable) costs associated with its orders, the holding costs incurred for its own inventories as well as the account management costs. The retailers are responsible for all costs incurred for orders they place with the supplier and all inventories they carry at their sites. (Later, we will show that the same analysis goes through if the supplier incurs a fixed order-processing cost for each retailer order.) Finally, all channel members place their replenishment orders at epochs chosen from the discrete set of power-of-two values $\{2^m T_L : m = \dots, -2, -1, 0, 1, 2, \dots\}$. This restriction results in considerable simplifications and efficiencies for the supplier at minimal expense to the retailers. As discussed before and known since Brown (1959), the latter's inventory related costs increase by at most 6%. On the other hand, if the retailers were able to choose their replenishment intervals without any restriction, these would be set according to the EOQ formula and the resulting order stream for the supplier would be highly nonstationary and in general without a periodically repeating pattern. This rather intricate order pattern from the retailers would certainly represent a difficult managerial task for the supplier. Moreover, no satisfactory solution exists for the resulting inventory problem for the supplier and the supplier's costs are significantly larger than if orders come in according to power-of-two patterns. It can therefore be expected that the supply-chain-wide profits in the Stackelberg game with the power-of-two restriction are larger than those in a version of the game without such restrictions. If the restriction to power-of-two intervals is difficult to enforce, the supplier may offer an annual rebate to the retailers equal to the modest increase in their inventory and setup costs resulting from the interval restriction. (Such rebates are most easily computed and clearly do not affect the supply-chain-wide profits arising from the Stackelberg game.)

We now develop an efficient algorithm for solving the above Stackelberg game. We begin by considering retailer i's problem. Let w be the wholesale price and T_0 the supplier's replenishment interval. Given these decisions, retailer i sets his own retail price p_i and replenishment interval T_i to maximize his own profit. Therefore, retailer i's problem can be formulated as

$$\max_{d_i, T_i} \pi_i(d_i, T_i|w) \stackrel{\text{def}}{=} (p_i(d_i) - c_i - w)d_i - \frac{K_i}{T_i} - \frac{1}{2}\bar{h}_i d_i T_i, \quad i = 1, \dots, N, \tag{13}$$

where T_i only takes on power-of-two values and d_i is confined to the finite interval $[d_i^{\min}, d_i^{\max}]$. (As before, we use d_i as the decision variable. The retail price is again uniquely determined through the inverse demand function.) Note that the profit measure in Equation (13) depends only on a single parameter specified by the supplier, i.e., the (constant) wholesale price w. In particular, the profit measure is independent of T_0.

We adopt the convention that when given a choice between two pairs of decision variables (d_i, T_i) with different T_i- but identical profit values, the retailer will always choose the pair with the lower T_i value. Note that for any fixed T_i value, the corresponding optimal d_i value is uniquely determined since the profit function π_i is strictly concave in d_i. Furthermore, this optimal d_i value is strictly decreasing in T_i, as follows from the first-order condition. Therefore, if there are multiple optimal solutions to the retailer's problem, they must have distinct d_i and T_i values. Thus, given the above convention, let $(d_i(w), T_i(w))$ be the (uniquely) chosen optimal solution in Equation (13).

Just like for any given T_i, there is a unique corresponding optimal d_i value, the converse holds as well, given the above convention. For any fixed d_i, the chosen optimal corresponding value of T_i is obtained by rounding the unconstrained minimum point $\sqrt{2K_i/\bar{h}_i d_i}$ of the EOQ cost function

$$\min_{T_i} \frac{K_i}{T_i} + \frac{1}{2}\bar{h}_i d_i T_i$$

to the power-of-two value that is closest in the following relative sense. Let $t_m = 2^m T_L$, m integer. The optimal value of T_i equals t_m if and only if

$$\frac{t_m}{\sqrt{2}} < \sqrt{\frac{2K_i}{\bar{h}_i d_i}} \leqslant t_m\sqrt{2} \quad \text{or} \quad d_i^m \leqslant d_i < d_i^{m-1}, \tag{14}$$

where $d_i^n = K_i/\bar{h}_i t_n^2$ for any integer n. Because d_i is confined to a finite interval $[d_i^{\min}, d_i^{\max}]$, there are only a finite number of T_i values that need to be considered. Let n_i be the number of such T_i values.

LEMMA 3. *For* $i = 1, \ldots, N$, *we have* (i) $d_i(w)$ *is decreasing in* w, (ii) $T_i(w)$ *is an increasing step function of* w *with finitely many steps, and* (iii) $T_i(w)$ *is still an increasing step function of* w *if* T_i *is confined to any finite subset of power-of-two values.*

PROOF. The rounding procedure in Equation (14) implies that the optimal value of T_i is uniquely determined by d_i and is decreasing in d_i. Moreover, as noted above, there are only a finite number of T_i values that are potentially optimal and thus the function $T_i(w)$ takes on finitely many values. Therefore, (i) implies (ii).

To show (i), take any $w' < w''$. Note that

$$\pi_i(d_i, T_i|w') - \pi_i(d_i, T_i|w'') = (w'' - w')d_i. \tag{15}$$

Write d_i' and T_i' for $d_i(w')$ and $T_i(w')$, respectively. Note that for any $d_i > d_i'$ and any T_i, we have

$$\begin{aligned}\pi_i(d_i', T_i'|w'') &= \pi_i(d_i', T_i'|w') - (w'' - w')d_i' \\ &> \pi_i(d_i', T_i|w') - (w'' - w')d_i \\ &= \pi_i(d_i, T_i|w''),\end{aligned}$$

where the inequality follows because $\pi_i(d_i', T_i'|w') \geqslant \pi_i(d_i, T_i|w')$ by the definition of d_i' and T_i', and $d_i > d_i'$. Therefore, $d_i(w'') \leqslant d_i'$.

Now suppose T_i is confined to a finite subset of power-of-two values. It is clear that in this case, the optimal value of T_i is still uniquely determined by d_i and decreases as d_i increases. (iii) thus follows from (i). □

Because $T_i(w)$ is an increasing step function of w with finitely many steps (Lemma 3), there exist a finite number of breakpoints for the wholesale price so that $T_i(w)$ remains constant between any two consecutive breakpoints. We now show how these breakpoints can be determined efficiently. First, note that we can restrict w to a finite interval (a, b) with $a = c_0$ and $b = \max\{p_i^{\max}, i = 1, \ldots, N\}$. Let $d_i(w, T_i)$ be the optimal demand rate for retailer i as a function of w and T_i. Let $\pi_i(w|T_i)$ denote $\pi_i(d_i(w, T_i), T_i|w)$. (Clearly, $\pi_i(w|T_i)$ is a decreasing function of w for any given T_i.) Now take any pair of reorder intervals T_i' and T_i'' with $T_i' < T_i''$, and suppose the retailer chooses between these two intervals. By Lemma 3 (iii), one of the following three cases prevails: (1) the retailer chooses T_i' for all $w \in (a, b)$; (2) the retailer chooses T_i'' for all $w \in (a, b)$; and (3) there exists a point w_0 such that T_i' is chosen to the left of w_0 and T_i'' is chosen to the right of w_0. Thus in all three cases, there exists a unique cross point $\hat{w}$ such that at this point, the retailer's choice switches permanently from T_i' to T_i'': in case (1), $\hat{w} = b$; in case (2), $\hat{w} = a$; in case (3), $\hat{w} = w_0$. Note that $\hat{w}$ can be determined easily by bisection, starting with the evaluation of the sign of $\pi_i(w|T_i') - \pi_i(w|T_i'')$ for $w = a$ and $w = b$.

Recall that there are n_i distinct power-of-two values of T_i that are potentially optimal. We construct the step function $\{T_i(w) : a < w < b\}$ by an iterative procedure. In the first iteration, we consider only the two smallest feasible power-of-two values, in the kth iteration the $(k+1)$ smallest values until reaching the $n_i - 1$st iteration in which all feasible power-of-two values are considered. By Lemma 3 (iii), if only the k smallest values of T_i are considered, the function $\{T_i(w) : a < w < b\}$ is still an increasing step function and can therefore be characterized by a (possibly empty) list of breakpoints $W_i = \{w_{(1)}, \ldots, w_{(L)}\}$ with $0 \leqslant L < k$ and an associated list of indices $J_i = \{j_{(1)}, \ldots, j_{(L)}\}$ such that for all $l = 1, \ldots, L$, $T_i(w)$ equals the $j_{(l)}$th smallest power-of-two value on the interval $(w_{(l-1)}, w_{(l)}]$ (with the convention $w_{(0)} = a$), while $T_i(w)$ equals the kth smallest power-of-two value on $(w_{(L)}, b]$. To simplify the exposition below, we write "$T_i = k$" when T_i equals the kth smallest power-of-two value. After the first iteration, both lists contain one element with $w_{(1)}$ the cross point of the profit functions associated with the two smallest power-of-two values. Assume now that the $k-1$th iteration has been completed. We describe how the two lists can be updated efficiently in the kth iteration. First, compute the cross point $\hat{w}$ of the profit functions of "$T_i = k$" and "$T_i = k+1$" and compare this with $w_{(L)}$. Case I: $\hat{w} > w_{(L)}$: set $L = L+1$, add $w_{(L)} = \hat{w}$ at the back of list W_i and add k at the back of J_i as $j_{(L)}$. (By the cross point definition, "$T_i = k+1$" dominates "$T_i = k$" for $w > \hat{w}$ while the list W_i indicates that "$T_i = k$" dominates all smaller power-of-two values for $w > \hat{w} > w_{(L)}$. Thus "$T_i = k+1$" dominates the k smallest values for $w > \hat{w}$, while one of the k smallest values dominates for $w \leqslant \hat{w}$. Indeed, for all $l = 1, \ldots, L$, "$T_i = j_{(l)}$" continues to dominate on the interval $(w_{(l-1)}, w_{(l)}]$.) Case II: $\hat{w} \leqslant w_{(L)}$: In this case, "$T_i = k$" is dominated by one of the other values throughout the interval (a, b) and can therefore be eliminated from consideration. Thus, delete $w_{(L)}$ and $j_{(L)}$ from the lists. The situation is now equivalent to that of iteration $j_{(L)}$, assuming all values $j_{(L)}+1, \ldots, k$ are eliminated from consideration. Thus, as before, compute the cross point $\hat{w}$ of the profit functions "$T_i = j_{(L)}$" and "$T_i = k+1$," set $L := L - 1$, and compare the cross point with the new $w_{(L)}$ value. Repeat this process, eliminating the elements from the back of the lists W_i and J_i and reducing the value of L by one until either Case I emerges or the lists are emptied out. After the n_ith iteration, the lists W_i and J_i, both of length $L \leqslant n_i - 1$, fully characterize the step function $\{T_i(w)\colon a < w < b\}$.

Note that the above algorithm bears similarity to other list-based procedures used to solve dynamic programs of special structure, see e.g., Aggarwal et al. (1987), Hirshberg and Larmore (1987), Wilber (1988), Miller and Myers (1988), Galil and Giancarlo (1989), and Federgruen and Tzur (1991).

Let $\bigcup_{i=1}^N W_i = \{w_0, w_1, \ldots, w_{M+1}\colon a = w_0 < w_1 < \cdots < w_M < w_{M+1} = b\}$ for some integer $M \leqslant \sum_{i=1}^N (n_i - 1)$ denote the complete list of all of the retailers' breakpoints. For all $w \in [w_m, w_{m+1}]$, the optimal replenishment intervals for the retailers remain fixed, $m = 0, \ldots, M$. Let T_i^m be retailer i's optimal replenishment interval when $w \in [w_m, w_{m+1}]$, for $m = 0, \ldots, M$ and $i = 1, \ldots, N$. Note that on the

m-th interval $[w_m, w_{m+1}]$, $d_i(w)$ is easily determined as follows:

$$d_i(w) = \begin{cases} d_i^{\min} & \text{if } \partial\pi_i(d_i^{\min}, T_i^m|w)/\partial d_i < 0, \\ d_i^{\max} & \text{if } \partial\pi_i(d_i^{\max}, T_i^m|w)/\partial d_i > 0, \\ \text{unique root of} & \\ \quad \partial\pi_i(d_i, T_i^m|w)/\partial d_i = 0 & \text{otherwise,} \end{cases}$$

where

$$\frac{\partial\pi_i(d_i, T_i^m|w)}{\partial d_i} = (p_i(d_i) - c_i - w) + p_i'(d_i)d_i - \frac{1}{2}\bar{h}_i T_i^m, \quad i = 1, \ldots, N.$$

Finally, it is noteworthy that each retailer's optimal profits vary continuously with the wholesale price w, in spite of the retailer's optimal replenishment interval being confined to a discrete set of values. More specifically:

Lemma 4. *$\pi_i(d_i(w), T_i(w)|w)$ is continuous and decreasing in w for $i = 1, \ldots, N$.*

Proof. Take any $i = 1, \ldots, N$. Because retailer i's profit under a higher wholesale price is lower for any d_i and T_i values, it follows that $\pi_i(d_i(w), T_i(w)|w)$ decreases in w. Now take any $w' < w''$. Write d_i', T_i', d_i'', and T_i'' for $d_i(w')$, $T_i(w')$, $d_i(w'')$, and $T_i(w'')$, respectively. Note that

$$\begin{aligned} 0 &\leqslant \pi_i(d_i', T_i'|w') - \pi_i(d_i'', T_i''|w'') \\ &= \pi_i(d_i', T_i'|w'') - \pi_i(d_i'', T_i''|w'') \\ &\quad + (w'' - w')d_i' \leqslant (w'' - w')d_i', \end{aligned}$$

where the first inequality follows because the retailer's optimal profits are decreasing in the wholesale price, the equality follows from Equation (15), and the second inequality follows from the definition of d_i'' and T_i''. Because d_i' is bounded, it follows that $\pi_i(d_i(w), T_i(w)|w)$ is a continuous function of w. □

We now proceed to consider the supplier's optimization problem, which can be written as

$$\max_{w, T_0} \pi_0(w, T_0) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \left\{ (w - c_0)d_i(w) - \Psi(d_i(w)) - \frac{1}{2}h_0 d_i(w)[T_0 - T_i(w)]^+ \right\} - \frac{K_0}{T_0}, \quad (16)$$

where T_0 takes on only power-of-two values. Let w^0 be the optimal wholesale price and T_0^0 the optimal replenishment interval for the supplier. The supplier's maximum profit is Π_0^0. Let $d_i^0 = d_i(w^0)$, $T_i^0 = T_i(w^0)$, and Π_i^0 retailer i's maximum profits, $i = 1, \ldots, N$.

Note that $\pi_0(w, T_0)$ is best optimized by restricting w sequentially to each of the $M+1$ intervals $[w_m, w_{m+1}]$, $m = 0, \ldots, M$. Let w^m and T_0^m be the optimal solution to Equation (16) when w is restricted to the mth interval, $[w_m, w_{m+1}]$. Note that

$$\pi_0(w, T_0) = \sum_{i=1}^{N} \left\{ \left(w - c_0 - \frac{1}{2}h_0[T_0 - T_i^m]^+ \right) d_i(w) - \Psi(d_i(w)) \right\} - \frac{K_0}{T_0}, \quad w_m \leqslant w \leqslant w_{m+1}.$$

We next show that only a limited number of power-of-two values need to be considered for T_0. Define

$$\begin{aligned} S^m &= \min\{T_i^m, i = 1, \ldots, N\}, \\ U^m &= \max\{T_i^m, i = 1, \ldots, N\}, \\ T^m &= \text{smallest power-of-two value greater than} \\ &\qquad \text{or equal to } \sqrt{2K_0/(h_0 \textstyle\sum_{i=1}^N d_i(w_{m+1}))}. \end{aligned}$$

Lemma 5. *$S^m \leqslant T_0^m \leqslant \max\{T^m, U^m\}$, $m = 0, \ldots, M$.*

Proof. The lower bound follows because $\pi_0(w, T_0)$ is increasing in T_0 for $T_0 \leqslant S^m$. To prove the upper bound, first note that for $T_0 > \max\{T^m, U^m\}$

$$\pi_0(w, T_0) = \sum_{i=1}^{N} \left\{ (w - c_0)d_i(w) - \Psi(d_i(w)) + \frac{1}{2}h_0 d_i(w)T_i^m \right\} - \left\{ \frac{1}{2}h_0 \sum_{i=1}^{N} d_i(w)T_0 + \frac{K_0}{T_0} \right\}.$$

Also note that $\sqrt{\frac{2K_0}{h_0 \sum_{i=1}^N d_i(w)}}$ is the minimum point of $\frac{1}{2}h_0 \sum_{i=1}^N d_i(w)T_0 + \frac{K_0}{T_0}$, and that for any $w \in [w_m, w_{m+1}]$,

$$\sqrt{\frac{2K_0}{h_0 \sum_{i=1}^N d_i(w)}} \leqslant \sqrt{\frac{2K_0}{h_0 \sum_{i=1}^N d_i(w_{m+1})}} \leqslant T^m,$$

where the first inequality follows because from Lemma 3, $d_i(w) \geqslant d_i(w_{m+1})$ for all $w \in [w_m, w_{m+1}]$. Therefore, $\pi_0(w, T_0)$ is decreasing in T_0 over the range $T_0 > \max\{T^m, U^m\}$. The upper bound follows. □

The supplier's problem on the interval $[w_m, w_{m+1}]$, $m = 0, \ldots, M$, can be solved by fixing T_0 at each of the power-of-two values between S^m and $\max\{T^m, U^m\}$ and maximizing $\pi_0(w, T_0)$ over w. Later, we will show that for an important special case, $\pi_0(w, T_0)$ is concave in w over $[w_m, w_{m+1}]$, allowing for a simple maximization.

We are now ready to present an algorithm for solving the Stackelberg game. The first step of the algorithm is to determine the breakpoints $\{a = w_0, w_1, \ldots, w_M, w_{M+1} = b\}$ and the optimal reorder intervals T_i^m for all retailers i and all intervals $[w_m, w_{m+1})$. The second step involves two loops. The outer loop iterates through the intervals $[w_m, w_{m+1})$, $m = 0, \ldots, M$. For each of these intervals, the inner loop investigates each of a finite number of distinct power-of-two values of T_0. For a given interval $[w_m, w_{m+1})$ and a given value of T_0, we optimize $\pi_0(w, T_0)$ over $w \in [w_m, w_{m+1})$. If π_0 is concave in w, this optimization is straightforward, e.g., by bisection. (Later, we will show that this is indeed the case when the demand functions are linear.) Otherwise, if π_0 fails to be concave in w, then a more general single-variable optimization technique is required. When these two loops are completed, we have an optimal solution to the Stackelberg game.

ALGORITHM II.

Step 1. (Determine $\{w_m\}_{m=0}^{M+1}$ with $w_0 = a$ and $w_{M+1} = b$ and T_i^m, $i = 1, \ldots, N$, $m = 0, 1, \ldots, M$)
For $i := 1, \ldots, N$ do
begin
  Set $w_{(1)}$ as the cross point of $\pi_i(\cdot|\text{“}T_i = 1\text{”})$ and $\pi_i(\cdot|\text{“}T_i = 2\text{”})$;
  $j_{(1)} := 1$; $W_i := \{w_{(1)}\}$; $J_i := \{j_{(1)}\}$; $L := 1$; $\hat{k} := 2$;
  For $k := 2, \ldots, n_i - 1$ do
  begin
    Set $\hat{w}$ as the cross point of $\pi_i(\cdot|\text{“}T_i = \hat{k}\text{”})$ and $\pi_i(\cdot|\text{“}T_i = k+1\text{”})$;
    While $\hat{w} \leqslant w_{(L)}$ do
    begin
      $\hat{k} := j_{(L)}$; Eliminate $j_{(L)}$ and $w_{(L)}$ from J_i and W_i;
      Set $\hat{w}$ as the cross point of $\pi_i(\cdot|\text{“}T_i = \hat{k}\text{”})$ and $\pi_i(\cdot|\text{“}T_i = k+1\text{”})$;
      $L := L - 1$;
    end;
    $L := L + 1$; $j_{(L)} := \hat{k}$; $w_{(L)} := \hat{w}$; Add $j_{(L)}$ to J_i, and $w_{(L)}$ to w_i;
    $\hat{k} := k + 1$;
  end;
end;
Merge the lists $\{W_i, i = 1, \ldots, N\}$ into a single list W;

Step 2. $\Pi_0^0 := 0$;
For $m := 0, \ldots, M$ do
begin
  For each power-of-two value of T_0 between S^m and $\max\{T^m, U^m\}$ do
  begin
    Determine the maximum x of $\pi_0(x, T_0)$ on the interval $[w_m, w_{m+1}]$;
    If $\pi_0(x, T_0) \geqslant \Pi_0^0$ then
    begin
      $\Pi_0^0 := \pi_0(x, T_0)$; $w^0 := x$; $T_0^0 := T_0$;
      For $i := 1, \ldots, N$ do $d_i^0 := d_i(x)$; $T_i^0 := T_i^m$;
      $\Pi_i^0 := \pi_i(d_i^0, T_i^0|x)$;
    end;
  end;
end.

We will measure the complexity of Algorithm II in terms of (i) elementary operations, (ii) the number of times a cross point needs to be computed of a pair of profit functions, and (iii) the number of times the continuous, single-variable function $\pi_0(x, T_0)$ is optimized on one of the intervals $[w_m, w_{m+1})$. Recall that a cross point can be determined with simple bisection evaluating the sign of a difference function $\pi_i(\cdot|T_i) - \pi_i(\cdot|T_i')$ for some pair of values T_i, T_i', Fix $i = 1, \ldots, N$. The effort to construct the lists W_i and J_i consists of $n_i - 1$ iterations. In each iteration, there is one cross point calculation followed by the addition of an element to each of the lists W_i and J_i and an increase of L (Case I) and possibly several cross point calculations resulting in eliminations of elements and a reduction of L (Case II). However, the total number of times Case II occurs across all $(n_i - 1)$ iterations is at most $n_i - 1$ itself, since each such case is uniquely associated with one of the distinct power-of-two values. The entire effort to construct the lists W_i and J_i for a given $i = 1, \ldots, N$ thus consists of $O(n_i)$ elementary operations and at most $2(n_i - 1)$ cross point calculations. To construct the complete set of breakpoints across all $i = 1, \ldots, N$ therefore involves $O(\sum_{i=1}^N n_i)$ elementary operations and at most $2\sum_{i=1}^N (n_i - 1)$ cross point calculations. An additional $O(N \log N + \sum_{i=1}^N n_i)$ operations is needed to merge the lists $\{W_i, i = 1, \ldots, N\}$ into a single list.

To assess the complexity associated with the second step of Algorithm II, let n_0 be the number of power-of-two values between $\min\{S^m\colon m = 0, \ldots, M\}$ and $\max\{T^m, U^m\colon m = 0, \ldots, M\}$. Thus the outer loop of the second step of Algorithm II is traversed $M \leqslant \sum_{i=1}^N (n_i - 1)$ times and the inner loop at most n_0 times for each value of m. Each time this inner loop is executed, we maximize $\pi_0(x, T_0)$ (a nonlinear, continuous, single-variable, closed-form function) over a finite interval of x. Therefore, the second step of Algorithm II involves doing this maximization at most $n_0 \sum_{i=1}^N (n_i - 1)$ times.

The overall effort for the algorithm thus consists of $O(N \log N + \sum_{i=1}^N n_i)$ elementary operations, $2\sum_{i=1}^N (n_i - 1)$ cross-point calculations, and at most $n_0 \sum_{i=1}^N (n_i - 1)$ maximizations of a continuous, closed-form, single-variable function. As mentioned, practical problems have $n_i \leqslant 8 (i = 0, 1, \ldots, N)$, resulting in an algorithm that requires at most $14N$ cross-point calculations, $56N$ single-variable function maximizations, as well as $O(N \log N)$ elementary operations.

REMARK 1. Suppose for each order by retailer $i, i = 1, \ldots, N$, the supplier incurs a fixed order-processing cost K_i^s and the retailer incurs a fixed setup cost K_i^r. In this case, the problem facing retailer i in the Stackelberg game is still characterized by Equation (13) with K_i replaced by K_i^r. Clearly, this problem has the same structure as before. Therefore, it is still true that there exist breakpoints of the wholesale price $\{w_m\}_{m=0}^{m+1}$ such that for all $w \in [w_m, w_{m+1}]$, $m = 0, \ldots, M$, the optimal replenishment intervals for the retailers remain fixed, i.e., $T_i(w) = T_i^m$, $i = 1, \ldots, N$. Note that the supplier's objective function is obtained by subtracting

$$\sum_{i=1}^N \frac{K_i^s}{T_i(w)}$$

from the objection function in Equation (15). This extra term, however, remains fixed for $w \in [w_m, w_{m+1}]$. Therefore, the supplier's problem on each interval $[w_m, w_{m+1}]$ has the same structure as before. In conclusion, Algorithm II still applies in this case.

Remark 2. For the important special case where each retailer is facing a linear demand function:

$$p_i(d_i) = a_i - b_i d_i, \quad i = 1, \ldots, N,$$

with a_i and b_i retailer-specific positive constants, maximizing $\pi_0(w, T_0)$ over $w \in [w_m, w_{m+1}]$ becomes straightforward. To see this, simply note that $d_i(w)$ is linear for $w \in [w_m, w_{m+1}]$. Therefore, if, e.g., $\Psi(\cdot)$ has the fixed-plus-linear form, then $\pi_0(w, T_0)$ is quadratic in w for $w_m \leqslant w \leqslant w_{m+1}$. This significantly reduces the computational effort for the second step of Algorithm II.

7. NUMERICAL EXAMPLES

This section presents numerical examples that were used to compare the centralized solution, the silo solution, and the Stackelberg solution.

We have evaluated two sets of examples, one with identical retailers and the other with nonidentical retailers. In all examples, we assumed $\Psi(d) = f + ed$ and $p_i(d_i) = a_i - b_i d_i$ for some positive constants f, e, a_i and $b_i, i = 1, \ldots, N$. The base case for the first set of examples has the following parameters: $N = 10, K_0 = 500, h_0 = 5, c_0 = 10, f = 10, e = 1$, and $K_i = 10, h_i = 1, c_i = 1, a_i = 100, b_i = 20$ for $i = 1, \ldots, N$. The base case for the second set is the same except

$$c_i = 1 + i/10, \quad \text{and} \quad b_i = \delta(20 + i/10), \quad i = 1, \ldots, N,$$

with $\delta = 1$. All the examples in a set were obtained from the base case of the set by varying one parameter at a time.

Both Algorithm I and Algorithm II are very efficient, solving each instance with $N = 10$ retailers in a fraction of a second (less than 0.02 secs) on a PC with a Pentium 450-MHz processor. Because the computational effort is effectively linear in N (under the above described assumptions), this implies that even instances with several hundred retailers can be solved in approximately one CPU second. Figures 1 and 2 summarize the results, where the percentage profit decrease is relative to the maximum

Figure 1. Identical retailers.

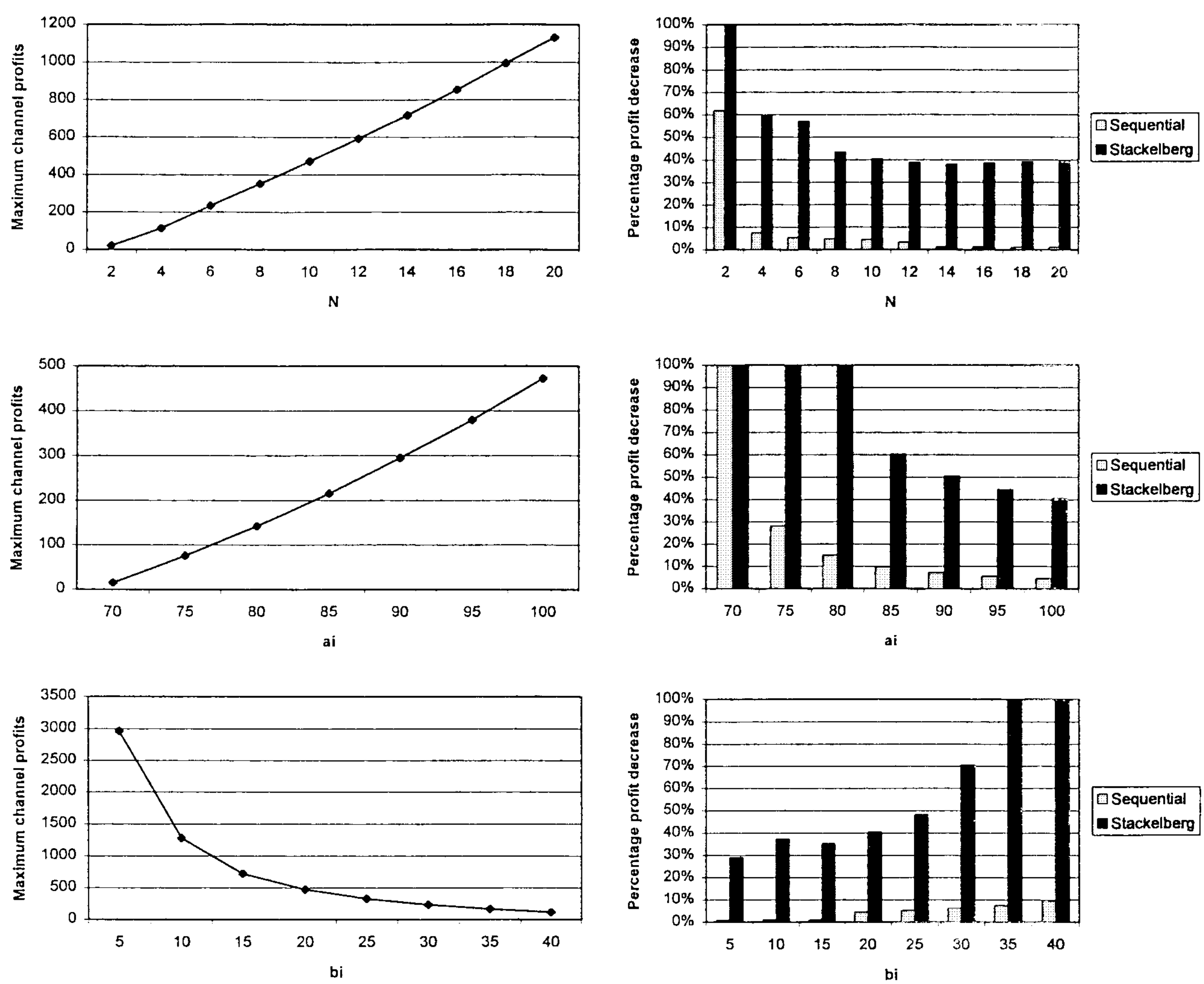
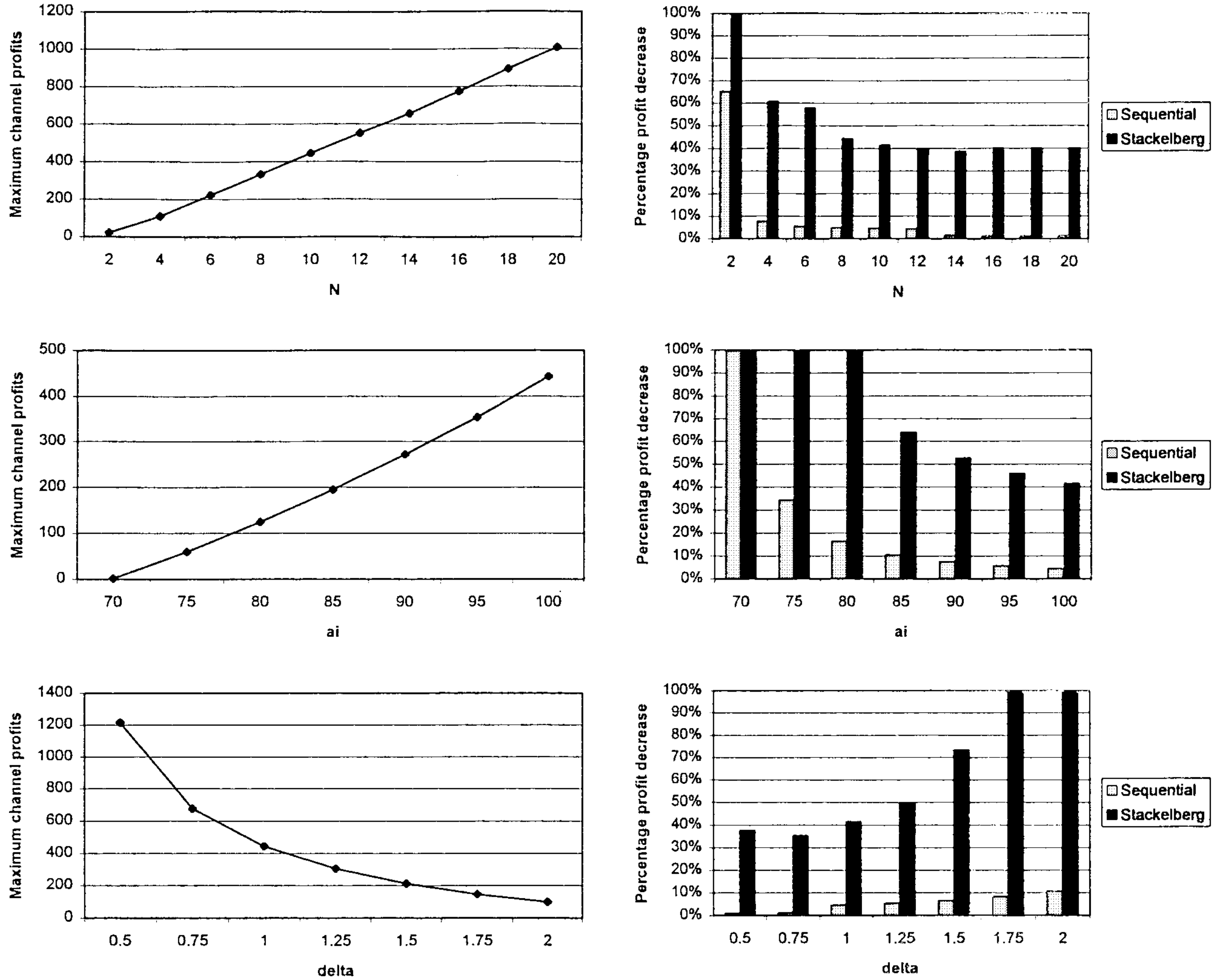


Figure 2. Nonidentical retailers.



channel profits (the joint optimum). Based on the results, we make the following observations.

• The system-wide profits decrease significantly in the Stackelberg game. In some cases, the supplier cannot ensure itself of a positive profit, even though a vertically integrated system (or an appropriately decentralized one, see Chen et al. 2001) results in healthy profits. In such settings, the supplier, in spite of being the price leader, is forced to withdraw from the market.

• Sequential (as opposed to joint) decision making can be quite costly, especially when the number of retailers is small or the price elasticity is large (a smaller value of a_i means larger price elasticity at any price level), or the slope of the inverse demand function (b_i) is large. (Because the demand function is $d_i = (a_i - p_i)/b_i$, a larger value of b_i means a smaller market at any given price.) In fact, we have identified instances where sequential decision making results in negative profits even though the maximum channel profits are positive. (In Figures 1 and 2, negative channel-wide profits are replaced by zero profits because in such cases the channel members withdraw from the market.) Note that the percentage profit decrease for small values of b_i (or δ) is consistent with an observation by Boyaci and Gallego (1997), who show that the relative profit decrease due to sequential decision making disappears as $b_i \to 0$.

• The economics literature (starting with Spengler 1950) has observed that wholesale prices arising in Stackelberg games exceed the marginal costs in a vertically integrated channel, resulting in higher retail prices and lower sales volumes. This phenomenon is usually referred to as double marginalization. We conjecture that this phenomenon is guaranteed to arise in our Stackelberg solution as well. While it is intrinsically difficult to prove this conjecture analytically, all our numerical results confirm the hypothesis. We observed in several instances that the sales volumes of the retailers in the Stackelberg solution are reduced by as much as 50% as compared to the centralized solution.

• It is often claimed that supply chain integration results in shorter replenishment cycle times, regarded by many as a benefit in and of itself. For example, Eckstut and Tang's (1996) recent survey of supply chain management practices in the pharmaceutical industry, observed that companies with explicit strategies to improve their supply chains have shorter cycle times both in manufacturing and replenishment as compared to others. Our results indicate that the mere integration or coordination of the decision making processes may fail to have the desired effect, by itself. As an example, in the base case with identical retailers, the retailers' replenishment intervals are 50% shorter in the Stackelberg solution as compared to the centralized solution. On the other hand, shorter cycle times are likely to arise if a coordinated or integrated decision making framework in the supply chain is exploited to achieve reductions in setup costs and setup times. Continuing with the same example, supply chain coordination must be accompanied by a reduction of the fixed costs associated with retailer orders (K_i) of at least 25% to achieve shorter cycle times.

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