# OPTIMAL POLICIES FOR MULTIECHELON INVENTORY PROBLEMS WITH MARKOV-MODULATED DEMAND

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This paper considers a multistage serial inventory system with Markov-modulated demand. Random demand arises at Stage 1, Stage 1 orders from Stage 2, etc., and Stage $N$ orders from an outside supplier with unlimited stock. The demand distribution in each period is determined by the current state of an exogenous Markov chain. Excess demand is backlogged. Linear holding costs are incurred at every stage, and linear backorder costs are incurred at Stage 1. The ordering costs are also linear. The objective is to minimize the long-run average costs in the system. The paper shows that the optimal policy is an echelon base-stock policy with state-dependent order-up-to levels. An efficient algorithm is also provided for determining the optimal base-stock levels. The results can be extended to serial systems in which there is a fixed ordering cost at stage $N$ and to assembly systems with linear ordering costs.

## 1. INTRODUCTION

This article is primarily concerned with the following serial production/distribution system. Customer demand arises at Stage 1, Stage 1 orders from Stage 2, etc., and Stage $N$ orders from an outside supplier with unlimited stock. The demand process is driven by an exogenous Markov chain, i.e., the state of the Markov chain in a period determines the demand distribution in that period. The objective is to minimize the long-run average holding and backorder costs in the system. We show that echelon base-stock policies with state-dependent order-up-to levels are optimal for the system. We also provide an efficient algorithm to compute an optimal policy.

Therefore, our model generalizes the seminal paper of Clark and Scarf (1960), who assume that customer demands in different periods are independent and identically distributed. The Markov-modulated demand process allows us to apply the model to a wide range of fluctuating demand environments attributable to, e.g., seasons, price changes, and economic conditions. The Clark-Scarf paper provides a theoretical foundation for much of the research in the increasingly important area of supply-chain management. We hope that the results presented herein will lead to the study of broader supply-chain issues.

There are other extensions of the Clark-Scarf model. Federgruen and Zipkin (1984) extend the optimality result to the infinite horizon case with either discounted or average costs. Schmidt and Nahmias (1985) characterize the optimal policy for an assembly system with two components. Rosling (1989) shows that a general assembly system is equivalent to a serial system and, thus, the Clark-Scarf result applies. Chen and Zheng (1994) provide simple optimality proofs for both the serial and assembly systems, which also hold for continuous-time models. All these papers, however, assume i.i.d. demands.

The demand model used here is not new to the single-location inventory literature; see, for example, Karlin (1960), Iglehart and Karlin (1962), Zipkin (1989), Ozekici and Parlar (1993), Song and Zipkin (1993), Aviv and Federgruen (1997), Beyer and Sethi (1997), Sethi and Cheng (1997), and Kapuscinski and Tayur (1998). The collective insight of these works is that the optimal policy for a model with fluctuating demand has the same structure as that in its stationary demand counterpart, except that the policy parameters must be adjusted to reflect the dynamics of the underlying demand environment. This paper shows that this insight can be carried over to multiechelon settings.

It is worth mentioning that other models of nonstationary demand have been used in the literature; see, for example, Johnson and Thompson (1975), Miller (1986), Lovejoy (1990, 1992), and Sobel (1997). These authors use various time-series models for the demand process. They all consider single-location models, except Sobel (1997). The focus is, again, to characterize the optimal policy. Note that these demand models invariably use internal information, such as past demands, to update the future demand distributions, whereas our demand process is driven by an exogenous random process.

Several multiechelon models incorporate nonstationary demand but focus on performance evaluation of a given policy or optimization within a given policy class, see, for example, Erkip et al. (1990), Song and Zipkin (1992, 1996), Drezner et al. (1996), Chen et al. (1997a, 1997b), and

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Baganha and Cohen (1998). Our goal here is, however, to characterize and compute the optimal policy.

The rest of the paper has three sections. For expositional purposes, we first consider a single-location model in §2. This allows us to clearly introduce the key ideas to be used later in the analysis of serial systems in §3. Section 4 describes extensions to systems with more general cost structures and to assembly systems.

## 2. SINGLE-LOCATION SYSTEMS

Consider the following inventory problem. Random demand arises periodically at a single location. Demand is satisfied from on-hand inventory with complete backlogging. Inventory is replenished from an outside source with ample stock. The replenishment lead time is constant: An order placed in period $t$ arrives in period $t+L$, where $L$ is a nonnegative integer. The cost incurred for an order is proportional to the order size. At the end of each period, holding costs are assessed based on the on-hand inventory at a constant rate of $h$ per unit, and penalty costs are assessed based on the backorders at a constant rate of $p$ per unit. The planning horizon is infinite. The objective is to minimize the long-run average cost in the system.

The demand process is driven by a discrete-time Markov chain $\mathbf{W} = \{W(t), t \geqslant 0\}$. It has $K$ states and is time homogeneous. Let $S = \{1, \ldots, K\}$ be the state space of $\mathbf{W}$ and $p_{kk'}$, $k$, $k' \in S$, be the one-step transition probability from $k$ to $k'$. We assume that $\mathbf{W}$ is ergodic. Therefore, the Markov chain has a unique stationary distribution, denoted by $\pi = (\pi_1, \ldots \pi_K)$. (If $\mathbf{W}$ is cyclic, i.e., $p_{k,k+1} = 1$ for $k < K$ and $p_{K,1} = 1$, then all the results hold by setting $\pi_k = 1/K, k \in S$.) We sometimes refer to $W(t)$ as the *demand state* in period $t$. If $W(t) = k$, then the demand in period $t$ is a nonnegative random variable denoted by $D_k$. Let $f_k(\cdot)$ be the probability mass function of $D_k$. (The demands in different periods with the same demand state are independent draws from the same probability distribution.) Given $W(t) = k$, let $D_k[t, t']$ be the total demand in periods $t, \ldots, t'$ and $D_k[t, t')$ the total demand in periods $t, \ldots, t'-1$.

As mentioned earlier, many authors have studied variations of the above model. Iglehart and Karlin (1962) study the discounted-cost version of the above model and show that a state-dependent base-stock policy is optimal. Song and Zipkin (1993) show that a state-dependent $(s, S)$ policy is optimal for a continuous-time, discounted-cost model with a fixed-order cost. This result is later extended by Sethi and Cheng (1997) to a discrete-time model with a more general cost structure (allowing, e.g., the fixed-order cost to be state-dependent) and by Beyer and Sethi (1997) to a discrete-time, average-cost model. Several authors have studied models where $\mathbf{W}$ is cyclic; see Karlin (1960), Zipkin (1989), Aviv and Federgruen (1997), and Kapuscinski and Tayur (1998). (The last two papers also incorporate capacity constraints on order quantities.)

The model considered in this section is a special case of the one in Beyer and Sethi (1997); their model allows fixed-order costs. In contrast to their dynamic programming approach, we use simple arguments to establish the optimality of state-dependent base-stock policies. We first establish a lower bound on the long-run average cost of any feasible policy through a series of demand-state reductions, and then show that the lower bound is reached by a state-dependent base-stock policy. This approach is appealing also because of its intuitive economic interpretations. Whereas Beyer and Sethi focus on the optimality proof, we also provide an efficient algorithm for computing the optimal base-stock levels. (Note that their proof requires a subtle assumption that the demand in each period is uniformly bounded; this is not necessary here.) Most importantly, the results here are essential to the analysis of serial systems in the next section. (The lower-bounding approach has led to various optimality results for other multiechelon systems; see Chen and Zheng 1994 and Chen 1999, 2000.)

For clarity, we assume that the events in each period occur in the following sequence. At the beginning of each period, (1) the state of the Markov chain $\mathbf{W}$ is observed; (2) a replenishment order, if any, is placed; and (3) a shipment is received from the outside supplier. Demand arrives during the period. At the end of the period, holding and backorder costs are assessed.

Let $IL(t)$ be the inventory level (on-hand inventory minus backorders) at the end of Period $t$. Define inventory position to be the total outstanding orders plus the inventory level. Let $IP(t)$ (resp., $IP^-(t)$) be the inventory position at the beginning of Period $t$ *after* (resp., *before*) ordering. Suppose $W(t) = k$. It is well known that

$$IL(t+L) = IP(t) - D_k[t, t+L]. \tag{1}$$

Notice that given $W(t) = k$, $IP(t)$ and $D_k[t, t+L]$ are independent. Because $IP(t)$ determines the expected holding and backorder costs in period $t+L$, it is reasonable to charge those costs to period $t$. Let $G^1(k, y)$ be the expected holding and backorder costs charged to period $t$ given $W(t) = k$ and $IP(t) = y$. From (1), we have

$$G^1(k, y) = E[h(y - D_k[t, t+L])^+ + p(D_k[t, t+L] - y)^+], \tag{2}$$

where $(x)^+ = \max\{x, 0\}$. Notice that this accounting scheme only shifts costs in time and therefore does not affect the long-run average cost of any policy. We call $G^1(k, \cdot)$ the *one-period cost for state $k$* under the current accounting scheme. (Because the long-run average order cost is a constant, which is the unit order cost times the long-run average demand for any feasible replenishment policy, we hereafter ignore the order cost.)

Take any feasible policy. We proceed to establish a lower bound on its long-run average cost. The procedure consists of $K$ iterations. At the beginning of each iteration $i$, a cost-accounting scheme is given. It specifies a partition of $S(=U^i \cup V^i)$ and a set of one-period costs $G^i(k, \cdot)$ for $k \in V^i$. The cost-accounting scheme charges $G^i(k, y)$ to period $t$ when $W(t)=k\in V^i$ and $IP(t)=y$, and charges zero cost to period $t$ when $W(t)\in U^i$. For convenience, we sometimes refer to the states in $U^i$ as *zero-cost states* and the states in $V^i$ as *cost-counting states.*

The first step in iteration $i\,(<K)$ is to identify a cost-counting state that will become a zero-cost state in the next iteration. Minimizing the one-period costs $G^i(k,y)$ over $y$ for all $k\in V^i$, we obtain $|V^i|$ minimum points, one for each state in $V^i$. Let $i^*$ be the state whose minimum point is the smallest among all the states in $V^i$; it will become a zero-cost state in the next iteration. The intuition for this choice is the following. The minimum point of $G^i(k,\cdot)$ is the *myopically* optimal base-stock level for any period with demand state $k$. Because the order cost is linear, one will never want to increase the inventory position to higher than the myopic level. However, for state $i^*$, the optimal order-up-to level should be equal to its myopic level, because it does not represent a constraint for the other periods that have the same or higher myopic levels.

The next step in iteration $i$ is to make state $i^*$ a zero-cost state by combining the one-period cost associated with $i^*$ with the one-period costs associated with the other states in $V^i$. To achieve this, first set $V^{i+1}=V^i\backslash\{i^*\}$ and $U^{i+1}=U^i\cup\{i^*\}$. For each period $t$ with demand state $k\in V^{i+1}$, charge a one-period cost, i.e., $G^{i+1}(k,\cdot)$, which, roughly speaking, represents the expected total cost incurred over an interval from period $t$ to the next time $\mathbf{W}$ comes back to $V^{i+1}$. (Therefore, $G^{i+1}(k,\cdot)$ represents the cost in a *mega period.*) Now we have a new cost-accounting scheme and are ready to proceed to the next iteration. Continue in this fashion until iteration $K$.

In the last iteration, there is only one cost-counting state and the corresponding inventory problem is essentially the same as the standard newsboy problem. The lower bound is obtained by solving this problem. In sum, the lower-bounding procedure effectively reduces the original inventory problem to a newsboy problem by successively reducing the number of cost-counting states.

We now describe the lower-bounding procedure in more detail. Recall that $S$ is partitioned into $U^i$ and $V^i$ in iteration $i$, $i=1,\dots,K$. Let $U^1=\varnothing$ and $V^1=S$. Formally, let

$G^i(k,y)=$ one-period cost charged to period $t$ in iteration $i$ if $W(t)=k\in V^i$ and $IP(t)=y$ for all $t$.

For convenience, set $G^i(k,\cdot)\equiv 0$ if $k\in U^i$ for each iteration $i$. Define

$C\{G(k,\cdot),k\in S\}=$ the long-run average cost when $G(k,y)$ is charged to period $t$ if $W(t)=k$ and $IP(t)=y$ for all $t$.

Note that $C\{G^1(k,\cdot),k\in S\}$ is the long-run average cost of the policy in question.

Consider iteration $i=1,\dots,K-1$. The cost-accounting scheme is specified by $U^i$, $V^i$, and $G^i(\cdot,\cdot)$. Suppose that $G^i(k,y)$, as a function of $y$, is convex and is minimized at a finite point $y=s^i(k)$, $k\in V^i$. This is clearly true for $i=1$ (see (2)). Define

$$i^*=\operatorname*{argmin}_{k\in V^i} s^i(k).$$

Define $\underline{G}^i(i^*,y)=G^i(i^*,\max\{y,s^i(i^*)\})$. Thus

$$G^i(i^*,y)\geqslant \underline{G}^i(i^*,y),\ \forall y. \tag{3}$$

Consequently,

$$C\{G^i(k,\cdot),k\in S\}\geqslant C\{G^i(k,\cdot),k\in S\backslash\{i^*\},\underline{G}^i(i^*,\cdot)\}. \tag{4}$$

In other words, if we replace the one-period cost for state $i^*$, $G^i(i^*,\cdot)$, with $\underline{G}^i(i^*,\cdot)$ and keep all the other one-period costs intact, then the long-run average cost does not increase. This new cost-accounting scheme is denoted by $\mathscr{C}^i$. Note that this step may lead to an undercharge of costs.

We now determine the cost-accounting scheme for the next iteration. Set $V^{i+1}=V^i\backslash\{i^*\}$ and $U^{i+1}=U^i\cup\{i^*\}$. It suffices to derive the one-period costs for the cost-counting states in $V^{i+1}$. Suppose $W(t)=k\in V^{i+1}$ and $IP(t)=y$ for some $t$. Let

$t(V)=$ the period before $\mathbf{W}$ comes back to $V^{i+1}$.

The one-period cost charged to period $t$ in iteration $i+1$, i.e., $G^{i+1}(k,y)$, is a lower bound on the expected total cost incurred in $[t,t(V)]$ under $\mathscr{C}^i$. That is

$$G^{i+1}(k,y)=\begin{cases}G^i(k,y) & W(t+1)\in V^{i+1}\\ G^i(k,y)+E[R^i(u,y-D_k)] & W(t+1)=u\in U^{i+1},\end{cases}$$

where

$R^i(u,z)=$ a lower bound on the expected total cost incurred in $[t+1,t(V)]$, given $W(t+1)=u\in U^{i+1}$ and $IP^-(t+1)=z$, under $\mathscr{C}^i$.

Equivalently,

$$G^{i+1}(k,y)=G^i(k,y)+\sum_{u\in U^{i+1}} p_{ku}E[R^i(u,y-D_k)]. \tag{5}$$

Therefore,

$$C\{G^i(k,\cdot),k\in S\backslash\{i^*\},\underline{G}^i(i^*,\cdot)\}\geqslant C\{G^{i+1}(k,\cdot),k\in S\}. \tag{6}$$

We now give the $R^i(\cdot,\cdot)$ functions. Suppose $W(t+1)=u\in U^{i+1}$ and $IP^-(t+1)=z$. Notice that in the interval $[t+1,t(V)]$, $\mathbf{W}$ evolves among the states in $U^{i+1}(=U^i\cup\{i^*\})$. Take any period $\tau$ in this interval. Because the order quantities are always nonnegative,

$$IP(\tau)\geqslant z-D_u[t+1,\tau). \tag{7}$$

Now, in the cost-accounting scheme $\mathscr{C}^i$, replace $IP(\tau)$ with $z-D_u[t+1,\tau)$ in charging costs to the period. When

$W(\tau) \in U^i$, the one-period cost charged to period $\tau$ is zero under $\mathscr{C}^i$ and thus the replacement has no impact on the cost. When $W(\tau) = i^*$, the one-period cost charged to period $\tau$ is $\underline{G}^i(i^*, IP(\tau))$ under $\mathscr{C}^i$, which is now replaced by $\underline{G}^i(i^*, z-D_u[t+1, \tau))$. Because $\underline{G}^i(i^*, \cdot)$ is nondecreasing in the inventory position, we have from (7)

$$\underline{G}^i(i^*, IP(\tau)) \geqslant \underline{G}^i\big(i^*, z-D_u(t+1, \tau)\big). \tag{8}$$

Therefore, this step may lead to another undercharge of costs. (The first such step is in (3).) Note that the above replacement of inventory positions is equivalent to assuming that no orders are placed in $[t+1, t(V)]$, which is intuitive because under $\mathscr{C}^i$, the one-period cost charged to each period in the interval is not decreasing in the inventory position. Then $R^i(u, z)$ is the expected total cost incurred in $[t+1, t(V)]$ after the replacement of inventory positions. We have

$$R^i(i^*, z) = \underline{G}^i(i^*, z) + \sum_{u\in U^{i+1}} p_{i^*u} E[R^i(u, z-D_{i^*})], \tag{9}$$

and

$$R^i(u, z) = \sum_{u'\in U^{i+1}} p_{uu'} E[R^i(u', z-D_u)], \quad u \in U^i. \tag{10}$$

To solve Equations (9) and (10), we first determine $R^i(u,z)$ for all $z \leqslant s^i(i^*)$ and $u \in U^{i+1}$. Take any $u \in U^{i+1}$ and $z \leqslant s^i(i^*)$. Take any period $\tau$ in the interval $[t+1, t(V)]$. If $W(\tau) \in U^i$, then the cost charged to the period is zero. Otherwise, if $W(\tau) = i^*$, then the cost charged is $\underline{G}^i(i^*, z-D_u[t+1, \tau)) = \underline{G}^i(i^*, s^i(i^*))$, which is independent of $z$, because $\underline{G}^i(i^*, x)$ is flat over $x \leqslant s^i(i^*)$ and $z \leqslant s^i(i^*)$. Therefore, $R^i(u,z)$ is also independent of $z$. Write $r^i(u)$ for $R^i(u,z)$ when $z \leqslant s^i(i^*)$. From (9) and (10),

$$r^i(i^*) = \underline{G}^i(i^*, s^i(i^*)) + \sum_{u\in U^{i+1}} p_{i^*u} r^i(u), \tag{11}$$

and

$$r^i(u) = \sum_{u'\in U^{i+1}} p_{uu'} r^i(u'), \quad u \in U^i. \tag{12}$$

Solving these linear equations, we obtain the constants $r^i(u), u \in U^{i+1}$. These constants can then be used in (9) and (10) to compute the entire functions $R^i(\cdot,\cdot)$ recursively. See the Appendix for more details.

LEMMA 1. *For any $k \in V^{i+1}$, $G^{i+1}(k, y)$, as a function of $y$, is convex and has a finite minimum point.*

PROOF. Take any $k \in V^{i+1}$. Suppose $W(t) = k$ and $IP(t) = y$. Let $\Omega$ be the set of all the sample paths of $\mathbf{W}$ that start from $k$. For each $\omega \in \Omega$, let $M(\omega)$ be the number of visits to state $i^*$ in $[t+1, t(V)]$. (Recall that $t(V)$ is the period before $\mathbf{W}$ comes back to $V^{i+1}$.) For $m = 1, \ldots, M(\omega)$, let $\mathscr{D}_m$ be the total demand in periods $t, t+1, \ldots, t_m - 1$, where $t_m$ is the period where the $m$th visit to state $i^*$ occurs. Recall that $G^{i+1}(k, y)$ is a lower bound on the expected total cost incurred in $[t, t(V)]$ under the cost-accounting scheme $\mathscr{C}^i$. It is obtained by assuming that no orders are placed in the interval. Therefore,

$$G^{i+1}(k,y) = G^i(k,y) + \sum_{\omega\in\Omega} Pr(\omega) \sum_{m=1}^{M(\omega)} E\underline{G}^i(i^*, y-\mathscr{D}_m). \tag{13}$$

Because both $G^i(k,\cdot)$ and $\underline{G}^i(i^*,\cdot)$ are convex because of the inductive assumption made at the beginning of iteration $i$, $G^{i+1}(k,\cdot)$ is convex. Moreover, because $G^i(k,\cdot)$ has a finite minimum point attributable to, again, the inductive assumption, and because $\underline{G}^i(i^*, x)$ is flat over $x \leqslant s^i(i^*)$, it follows that $G^{i+1}(k,\cdot)$ has a finite minimum point. □

In iteration $K$, $V^K$ contains a single state denoted by $K^*$. Let $s^K(K^*)$ be the minimum point of $G^K(K^*, \cdot)$. A lower bound on $C\{G^K(k,\cdot), k \in S\}$ is obtained by charging $G^K(K^*, s^K(K^*))$ to period $t$ when $W(t) = K^*$ and charging zero cost otherwise. Under this cost-accounting scheme, the long-run average cost is

$$B^* = \frac{1}{\mu_{K^*}} G^K(K^*, s^K(K^*)),$$

where $\mu_{K^*}$ is the expected time between two consecutive visits to state $K^*$ by $\mathbf{W}$. Note that $\mu_{K^*} = 1/\pi_{K^*}$ (see, for example, Ross 1970). This, together with (4) and (6), leads to the following theorem.

THEOREM 1. $C\{G^1(k,\cdot), k\in S\} \geqslant B^* = \pi_{K^*} G^K(K^*, s^K(K^*))$.

LEMMA 2. $s^1(1^*) \leqslant s^2(2^*) \leqslant \cdots \leqslant s^K(K^*)$.

PROOF. Take any $i = 1, \ldots, K-1$. It suffices to show that $s^i(i^*) \leqslant s^{i+1}(k)$ for all $k \in V^{i+1}$. Take any $k \in V^{i+1}$. Because $k$ is also a member of $V^i$, $s^i(k) \geqslant s^i(i^*)$ by the definition of $s^i(i^*)$. Thus the convex functions on the righthand side of (13) are nonincreasing over $y \leqslant s^i(i^*)$. Therefore, $s^{i+1}(k) \geqslant s^i(i^*)$. □

Let $s^*(k) = s^i(i^*)$, where $i^* = k$, $k \in S$. Consider the following state-dependent base-stock policy $\{s^*(k), k \in S\}$: for any period $t$, if $W(t) = k$, then follow a base-stock policy with order-up-to level $s^*(k)$. That is, place an order to increase the inventory position to $s^*(k)$ when it is below $s^*(k)$ and do nothing otherwise. This policy is optimal, as the following theorem shows.

THEOREM 2. *The state-dependent base-stock policy $\{s^*(k), k \in S\}$ is optimal.*

PROOF. Suppose the above policy is followed. It suffices to show that the long-run average cost in the system is equal to the lower bound established in Theorem 1. This is achieved by following the lower-bounding procedure developed earlier and showing that it does not cause any undercharge of costs. Recall that the procedure consists of $K$ iterations. In iteration $i$, $i = 1, \ldots, K-1$, there are two steps that may result in an undercharge of costs.

The first step is the replacement of $G^i(i^*, \cdot)$ with $\underline{G}^i(i^*, \cdot)$. This only affects the periods $t$ with $W(t) = i^*$. Take any such period $t$. Notice that under the above policy, we have $IP(t) = \max\{s^i(i^*), IP^-(t)\} \geqslant s^i(i^*)$. By the definition of $\underline{G}^i(i^*, \cdot)$, $\underline{G}^i(i^*, IP(t)) = G^i(i^*, IP(t))$. Therefore, the inequality in (4) becomes an equality.

The second step that may result in an undercharge of costs is in the determination of the one-period cost charged to period $t$ with $W(t) = k \in V^{i+1}$ in iteration $i+1$. In deriving the $R^i(u, z)$ functions, $u \in U^{i+1}$, we replaced $IP(\tau)$ with $z - D_u[t+1, \tau)$ for all the periods $\tau$ in the interval $[t+1, t(V)]$. If $W(\tau) \in U^i$, then this replacement has no impact on the cost because the cost-accounting scheme $\mathscr{C}^i$ charges zero cost to the period, regardless of the inventory position. Now suppose $W(\tau) = i^*$. There are two cases. If no orders are placed in periods $t+1, \ldots, \tau$, then $IP(\tau) = z - D_u[t+1, \tau)$. In this case, the inequality in (8) becomes an equality. On the other hand, if an order is placed in those periods, then the inventory position after that order must be equal to $s^l(l^*)$ for some $l = 1, \ldots, i$, which is less than or equal to $s^i(i^*)$ (Lemma 2). Therefore, $IP^-(\tau) \leqslant s^i(i^*)$. As a result, an order is placed to increase the inventory position to the corresponding base-stock level, and thus $IP(\tau) = s^i(i^*)$. From (7) and because $\underline{G}^i(i^*, y)$ is flat for $y \leqslant s^i(i^*)$, the inequality in (8) is still an equality. In sum, the second step does not result in an undercharge of costs either, and thus (6) becomes an equality.

In the last iteration, a lower bound is obtained by replacing $IP(t)$ with $s^K(K^*)$ if $W(t) = K^*$. However, under the above policy, $IP(t) = s^K(K^*)$ in the long run for all the periods $t$ with $W(t) = K^*$, because state $K^*$ has the maximal order-up-to level (Lemma 2). The theorem then follows. □

We conclude this section with the following algorithm for computing the optimal base-stock levels. Note that when **W** is cyclic, the algorithm reduces to the one in Zipkin (1989) and that Theorem 2 represents an alternative proof of his result for the infinite-horizon, average-cost case.

SINGLE-LOCATION ALGORITHM.

*Initialize*
$V^1 = S, U^1 = \varnothing$.
Set $i = 1$.

*Step 1*
Minimize $G^i(k, y)$ over $y, k \in V^i$. Let $s^i(k)$ be the minimum point.
$i^* = \text{argmin}_{k \in V^i} s^i(k)$.
$s^*(i^*) = s^i(i^*)$.
If $i < K$, go to *Step 2*. Otherwise, stop.

*Step 2*
$V^{i+1} = V^i \setminus \{i^*\}, U^{i+1} = U^i \cup \{i^*\}$.
Compute $R^i(u, \cdot)$ recursively for all $u \in U^{i+1}$.
Use (5) to compute $G^{i+1}(k, \cdot)$ for all $k \in V^{i+1}$.
$i \leftarrow i+1$. Go to *Step 1*.

## 3. SERIAL SYSTEMS

Consider the following serial inventory system with $N$ stages. Customer demand arises periodically at Stage 1, Stage 1 orders from Stage 2, 2 from 3, etc., and Stage $N$ orders from an outside supplier with unlimited stock. (The outside supplier is also called stage $N+1$). When demand exceeds the on-hand inventory at Stage 1, the excess is backlogged. The production-transportation leadtime from Stage $n+1$ to Stage $n$ is $L_n$ periods, where $L_n$ is a nonnegative constant. The system incurs linear holding costs for on-hand inventories at every stage and linear penalty costs for backorders at Stage 1. The objective is to minimize the long-run average total cost in the system.

The customer demand process is the same as in the single-location model considered in the previous section. It is driven by the $K$-state Markov chain **W** with a stationary distribution $\pi = (\pi_1, \ldots, \pi_K)$. (As before, when **W** is cyclic, all the results here hold by simply setting $\pi_k = 1/K$ for all $k \in S$.) Recall that given $W(t) = k$, $D_k$ is the customer demand in period $t$, $D_k[t, t')$ is the total customer demand in periods $t, \ldots, t'-1$, and $D_k[t, t']$ is the total customer demand in periods $t, \ldots, t'$.

The system has the following cost parameters:

$H_n =$ installation holding cost at Stage $n$ per unit per period,
$h_n =$ echelon holding cost at Stage $n$ per unit per period,
$\quad = H_n - H_{n+1} > 0$, with $H_{N+1} = 0$,
$p =$ backorder penalty cost (at Stage 1) per unit per period, $p > 0$.

Define the *echelon inventory level* at Stage $n$ to be the inventories on hand at Stages $1, \ldots, n$ plus inventories in transit to Stages $1, \ldots, n-1$ minus backorders at Stage 1. In short, the echelon inventory level at Stage $n$ is the *net* inventory level in the subsystem consisting of Stages $1, \ldots, n$. Define the *echelon inventory position* at Stage $n$ to be the echelon inventory level at Stage $n$ plus inventories in transit to Stage $n$.

As in the single-location model, we assume that all the replenishment activities in a period occur at the beginning of the period after observing the demand state. At Stage $n$ $(>1)$, the replenishment activities occur in the following sequence: An order from Stage $n-1$ is received, an order is placed with Stage $n+1$, a shipment is received from Stage $n+1$, and a shipment is sent to Stage $n-1$. For Stage 1, order placement occurs at the beginning of the period, whereas customer demand arrives during the period.

For Stage $n$ in period $t$, let

$IP_n^-(t) =$ echelon inventory position *before* ordering,
$IP_n(t) =$ echelon inventory position *after* ordering,
$IL_n^-(t) =$ echelon inventory level *before* demand,
$IL_n(t) =$ echelon inventory level *after* demand.

Let $B(t)$ be the backorder level at Stage 1 at the end of period $t$.

We follow the convention of assessing holding and backorder costs based on the period-ending inventory levels. Note that the installation (on-hand) inventory at Stage $n$ in period $t$, $n \geqslant 2$, can be written as $IL_n(t) - IL_{n-1}(t)$ and the installation (on-hand) inventory at stage 1 in period $t$ as $IL_1(t) + B(t)$. (Thus, the inventories in transit to Stage $n-1$ are part of the installation inventory of Stage $n$. This is standard.) Charging $H_n$ for each unit of installation inventory at Stage $n$, $n = 1, \ldots, N$, and $p$ for each unit of customer backorders, we have the following holding and backorder costs in period $t$:

$$\sum_{n=2}^{N} H_n[IL_n(t) - IL_{n-1}(t)] + H_1[IL_1(t) + B(t)] + pB(t) = \sum_{n=1}^{N} h_n IL_n(t) + (p + H_1)B(t). \tag{14}$$

Take any period $t$. Suppose $W(t) = k$. Note that $IL_1(t + L_1) = IP_1(t) - D_k[t, t + L_1]$, and $IP_1(t)$ and $D_k[t, t + L_1]$ are independent. Thus, given $IP_1(t) = y$, the expected value of $h_1 IL_1(t + L_1) + (p + H_1)B(t + L_1)$ is equal to

$$G_1^1(k, y) \stackrel{\text{def}}{=} E[h_1(y - D_k[t, t + L_1]) + (p + H_1)(D_k[t, t + L_1] - y)^+]. \tag{15}$$

Now charge the following costs in period $t$

$$\sum_{n=2}^{N} h_n IL_n(t) + G_1^1(W(t), IP_1(t)). \tag{16}$$

Clearly, both (14) and (16) lead to the same long-run, average systemwide cost.

For the rest of this section, we first establish a lower bound on the long-run average systemwide cost of any feasible policy and then construct a policy that achieves the lower bound. The lower bound is obtained by solving a sequence of single-location problems. We start at Stage 1 and decompose its one-period cost for each demand state to two components: One remains at Stage 1, and the other is charged to Stage 2. The costs that remain at Stage 1 lead to a problem with essentially the same structure as the one in the last section. This enables us to establish a lower bound on the remaining costs at Stage 1. At this time, Stage 1 can be eliminated, and the serial system has one fewer stage. The resulting $(N-1)$-stage serial system can be treated as similar to the original one, etc. In short, our lower-bounding method follows the idea of Clark Scarf (1960)—of charging induced penalty costs to the upstream stages.

Take any feasible policy. Let $\mathbf{X}(t)$ be a vector of inventory variables (e.g., inventory positions and/or levels) associated with period $t$. Define $C\{G(W(t), \mathbf{X}(t))\}$ to be the long-run average cost in a system in which $G(W(t), \mathbf{X}(t))$ is the one-period cost charged to period $t$ for all $t$.

For $m = 1, \ldots, N$, suppose the costs charged to period $t$ have the following form:

$$\sum_{n=m+1}^{N} h_n IL_n(t) + G_m^1(W(t), IP_m(t)), \tag{17}$$

where $G_m^1(k, y)$ is convex in $y$ and has a finite minimum point (over $y$), $k \in S$. This is clearly true for $m = 1$; see (15) and (16).

We next derive a lower bound on $C\{G_m^1(W(t), IP_m(t))\}$ by following the procedure used for the single-location model. There are $K$ iterations. Recall that at the beginning of iteration $i$, $i = 1, \ldots K$, a cost-accounting scheme is specified. Let it be $U_m^i$, $V_m^i$, and $G_m^i(\cdot, \cdot)$ with $V_m^1 = S$ and $U_m^1 = \varnothing$. (The subscript is usually reserved for stage and the superscript for iteration.) The first step in iteration $i$ is to minimize the convex functions $G_m^i(k, y)$ over $y$ for all $k \in V_m^i$. Let

$$s_m^i(k) = \text{the minimum point of } G_m^i(k, \cdot)$$
$$i_m^* = \operatorname*{argmin}_{k \in V_m^i} s_m^i(k)$$
$$s_m^*(i_m^*) = s_m^i(i_m^*).$$

Now decompose the one-period cost for state $i_m^*$, $G_m^i(i_m^*, y)$ into the following two components:

$$\underline{G}_m^i(i_m^*, y) = G_m^i(i_m^*, \max\{y, s_m^*(i_m^*)\}),$$

and

$$G_{m,m+1}(i_m^*, y) = G_m^i(i_m^*, y) - \underline{G}_m^i(i_m^*, y) = G_m^i(i_m^*, \min\{y, s_m^*(i_m^*)\}) - G_m^i(i_m^*, s_m^*(i_m^*)).$$

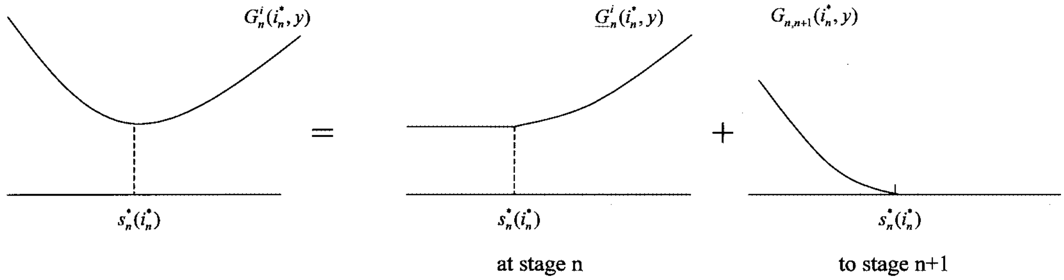
These cost components have clear economic interpretations. Suppose $s_m^*(i_m^*)$ is the optimal order-up-to level for the echelon inventory position at Stage $m$ for any period $t$ with $W(t) = i_m^*$. (This is indeed true, as we shall see later.) Take any period $t$ with $W(t) = i_m^*$. Note that $\underline{G}_m^i(i_m^*, IP_m(t))$ is constant in $IP_m(t)$ until it exceeds $s_m^*(i_m^*)$, which happens only when a large order is placed by Stage $m$ before $t$ (under a different demand state). In other words, this cost component should be Stage $m$'s responsibility. On the other hand, $G_{m,m+1}(i_m^*, IP_m(t)) > 0$ only when $IP_m(t) < s_m^*(i_m^*)$, which occurs only when the upstream stage has insufficient on-hand inventory. Therefore, this cost component should be charged to Stage $m+1$. Notice that in the single-location model, we discarded $G_{m,m+1}(i_m^*, \cdot)$ in establishing the lower bound. Now it is "recycled" by charging it to the upstream stage. (When $m = N$, the second component is effectively discarded, because stage $N+1$ is the outside supplier.)

Recall that $G_m^i(i_m^*, y)$ is convex in $y$ (Lemma 1). Thus both $\underline{G}_m^i(i_m^*, y)$ and $G_{m,m+1}(i_m^*, y)$ are convex in $y$. Moreover, the former is nondecreasing and the latter nonincreasing in $y$. See Figure 1 for an illustration of these two functions. Because $IP_m(t) \leqslant IL_{m+1}^-(t)$ by definition,

$$G_{m,m+1}(i_m^*, IP_m(t)) \geqslant G_{m,m+1}(i_m^*, IL_{m+1}^-(t)), \tag{18}$$

with $IL_{N+1}(t) = \infty$. Now charge $G_{m,m+1}(i_m^*, IL_{m+1}^-(t))$ to Stage $m+1$ for all periods $t$ with $W(t) = i_m^*$. When $i = K$, the procedure ends. Otherwise, let $V_m^{i+1} = V_m^i \backslash \{i_m^*\}$ and $U_m^{i+1} = U_m^i \cup \{i_m^*\}$. Determine $G_m^{i+1}(k, \cdot)$, $k \in V_m^{i+1}$, as in the

**Figure 1.** Cost allocation at Stage $n$ for iteration $i$.




previous section. Let $G_m^{i+1}(k, y) \equiv 0$, $k \in U_m^{i+1}$. Then proceed to iteration $i+1$. In the last iteration, we obtain

$$B_m^* \overset{\text{def}}{=} \pi_{K_m^*} G_m^K(K_m^*, s_m^*(K_m^*)).$$

From the previous section and with (18), we have

$$C\{G_m^1(W(t), IP_m(t))\} \geqslant C\{G_{m,m+1}(W(t), IL_{m+1}^-(t))\} + B_m^*.$$

Therefore, the long-run average value of (17) is greater than or equal to the long-run average value of

$$\sum_{n=m+1}^{N} h_n IL_n(t) + G_{m,m+1}(W(t), IL_{m+1}^-(t)) + B_m^*. \tag{19}$$

Now we derive an alternative form of (19) without changing its long-run average value. Notice that given $W(t) = k$,

$$IL_{m+1}(t + L_{m+1}) = IP_{m+1}(t) - D_k[t, t + L_{m+1}],$$

and

$$IL_{m+1}^-(t + L_{m+1}) = IP_{m+1}(t) - D_k[t, t + L_{m+1}).$$

Moreover, $IP_{m+1}(t)$ is independent of $D_k[t, t+L_{m+1}]$ and $D_k[t, t+L_{m+1})$. For $k \in S$, define

$$\begin{aligned} G_{m+1}^1(k, y) &= E[h_{m+1} IL_{m+1}(t+L_{m+1}) \\ &\qquad + G_{m,m+1}(W(t+L_{m+1}), IL_{m+1}^-(t+L_{m+1})) \\ &\qquad | W(t)=k, IP_{m+1}(t)=y] \\ &= E[h_{m+1}(y - D_k[t, t+L_{m+1}]) \\ &\qquad + G_{m,m+1}(W(t+L_{m+1}), y - D_k[t, t+L_{m+1})) \\ &\qquad | W(t)=k]. \end{aligned}$$

Thus $G_{m+1}^1(k, y)$ is convex in $y$. Also, the function has a finite minimum point. To see this, it suffices to have $\lim_{y\to\pm\infty} G_{m+1}^1(k, y) = \infty$. This follows by noting that $G_{m,m+1}(k, y)$, as a function of $y$, has an asymptotic slope of $-(p + H_{m+1})$ as $y \to -\infty$ and an asymptotic slope of zero as $y \to +\infty$. Therefore, $G_{m+1}^1(k, y)$, as a function of $y$, has an asymptotic slope of $-(p + H_{m+1}) + h_{m+1} = -(p + H_{m+2}) < 0$ as $y \to -\infty$ and an asymptotic slope of $h_{m+1}$ as $y \to +\infty$. We omit the details for brevity. Note that the long-run average value of (19) is equal to the long-run average value of

$$\sum_{n=m+2}^{N} h_n IL_n(t) + G_{m+1}^1(W(t), IP_{m+1}(t)) \tag{20}$$

plus the constant $B_m^*$. Because (20) has exactly the same form as (17), one can repeat the above procedure to determine a lower bound on $C\{G_{m+1}^1(W(t), IP_{m+1}(t))\}$. Continue in this fashion until the lower bound on $C\{G_N^1(W(t), IP_N(t))\}$, i.e., $B_N^*$, is determined.

REMARK. There is a recursive procedure for computing the $G_{m+1}^1(\cdot, \cdot)$ functions. First, define

$$H_{m+1}(k, y; l) = E[G_{m,m+1}(W(t+l), y - D_k[t, t+l)) \mid W(t) = k].$$

Thus,

$$G_{m+1}^1(k, y) = E[h_{m+1}(y - D_k[t, t+L_{m+1}])] + H_{m+1}(k, y; L_{m+1}). \tag{21}$$

Note that

$$H_{m+1}(k, y; 0) = G_{m,m+1}(k, y),$$

and that

$$H_{m+1}(k, y; l+1) = \sum_{k' \in S} p_{kk'} E[H_{m+1}(k', y - D_k; l)].$$

THEOREM 3. *$\sum_{n=1}^{N} B_n^*$ is a lower bound on the long-run average cost of any feasible policy in the serial system.*

In the process of constructing the lower bound in Theorem 3, we have determined a set of critical numbers for each stage, i.e., $\{s_n^*(k), k \in S\}_{n=1}^N$. Now consider the following replenishment policy. For any period $t$ with $W(t) = k \in S$, if the echelon inventory position at Stage $n$, $IP_n^-(t)$, is below $s_n^*(k)$, then Stage $n+1$ sends a shipment to Stage $n$ to increase its echelon inventory position to $s_n^*(k)$ or if Stage $n+1$ has insufficient on-hand inventory, ship as much as possible, $n = 1, \ldots, N$. When $IP_n^-(t)$ is above $s_n^*(k)$, do not ship. (Recall that the outside supplier, i.e., Stage $N+1$, has ample stock. Thus Stage $N$ is always able to increase its echelon inventory position to the corresponding base-stock level.) This echelon base-stock policy actually achieves the lower bound in Theorem 3, as the following theorem shows.

THEOREM 4. *The echelon base-stock policy* $\{s_n^*(k), k \in S\}_{n=1}^N$ *is optimal for the serial system.*

PROOF. It suffices to show that the long-run average cost of the above policy is equal to the lower bound in Theorem 3. We first show that (17) and (19) have the same long-run average value under the policy for $m = 1, \ldots, N-1$. Consider iteration $i$, $i = 1, \ldots, K$. Recall that in this iteration, we first identify a state, i.e., $i_m^*$, in $V_m^i$. Whenever $W(t) = i_m^*$, we split the one-period cost $G_m^i(i_m^*, \cdot)$ into two parts. One remains at Stage $m$, and the other is charged to Stage $m+1$. This step may result in an undercharge of costs attributable to the inequality in (18). However, this inequality is in fact an equality under the policy. To see this, first recall that $IP_m(t) < IL_{m+1}^-(t)$ by definition. Now consider the following two cases.

*Case 1.* $IL_{m+1}^-(t) < s_m^*(i_m^*)$. Because Stage $m$ follows the echelon base-stock policy with order-up-to level $s_m^*(i_m^*)$ whenever $W(t) = i_m^*$, we have $IP_m(t) = IL_{m+1}^-(t)$ (this corresponds to the scenario in which Stage $m+1$ ships all of its on-hand inventory, if any, to Stage $m$).

*Case 2.* $IL_{m+1}^-(t) \geqslant s_m^*(i_m^*)$. In this case, again attributable to the base-stock policy, $IP_m(t) \geqslant s_m^*(i_m^*)$. However, $G_{m,m+1}(i_m^*, y)$ is flat for $y \geqslant s_m^*(i_m^*)$. Therefore, in both cases, the inequality in (18) is an equality.

There is another step in iteration $i$ in which undercharging of costs may occur, i.e., the determination of the one-period costs for the states in $V_m^{i+1}$. One can show that here, again, there is no undercharging under the policy by following the proof of Theorem 2, with a few minor changes. Recall from the previous section that the determination of the one-period costs is based on the replacement of $IP_m(\tau)$ with $z - D_u[t+1, \tau)$, given $W(t+1) = u \in U_m^{i+1}$ and $IP_m^-(t+1) = z$, for all the periods $\tau$ in the interval $[t+1, t(V)]$. If $W(\tau) \neq i_m^*$, then the replacement has no impact on costs because the one-period costs for the demand states in $U_m^i$ are zero, independent of the inventory position. Now suppose $W(\tau) = i^*$. There are two cases. If no shipments are sent to Stage $m$ in periods $t+1, \ldots, \tau$, then $IP_m(\tau) = z - D_u[t+1, \tau)$. Thus, the inequality in (8) is an equality. On the other hand, if a shipment is sent in those periods, then Stage $m$'s echelon inventory position after that shipment must be less than or equal to $s_m^l(l_m^*)$ for some $l = 1, \ldots, i$. From Lemma 2, this inventory position is less than or equal to $s_m^*(i_m^*)$. Therefore, $IP_m(\tau) \leqslant s_m^i(i_m^*)$. From (7) and the fact that $\underline{G}_m^i(i_m^*, y)$ is flat for $y \leqslant s_m^i(i_m^*)$, the inequality in (8) is still an equality.

When $m = N$, (17) corresponds to a single-location problem. By Theorem 2, the state-dependent base-stock policy for Stage $N$ achieves the lower bound.

We conclude with the following algorithm.

MULTISTAGE ALGORITHM.

*Initialization*
Compute $G_1^1(k, \cdot)$, $k \in S$.

*Main Loop*
For $m = 1, \ldots, N$ do
begin
$G^1(k, \cdot) = G_m^1(k, \cdot)$, $k \in S$.
Go to the single-location algorithm.
$s_m^*(k) = s^*(k)$, $k \in S$.
if $m < N$, then
begin
$G_{m,m+1}(i^*, y) = G^i(i^*, y) - \underline{G}^i(i^*, y)$, $i = 1, \ldots, K$.
Use (21) to compute $G_{m+1}^1(k, \cdot)$, $K \in S$.
end.
end.

## 4. CONCLUDING REMARKS

Some extensions are possible. For example, it is easy to incorporate a fixed ordering cost at Stage $N$. In this case, the optimal policy at Stage $N$ is an echelon $(s, S)$ policy with state-dependent control parameters. This follows from Beyer and Sethi (1997), because the lower-bounding procedure eventually reduces the multistage problem to a single-stage one. To generalize the results to assembly systems, one can follow Rosling (1989) by showing the equivalence between assembly and serial systems. The detailed proof can be obtained by mimicking Chen and Zheng (1994).

## APPENDIX

The appendix can be found at the *Operations Research* Home Page in the Online Collection: ⟨http://or.pubs.informs.org⟩.

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