Sales-Force Incentives and Inventory Management

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This article studies the problem of sales-force compensation by considering the impact of sales-force behavior on a firm's production and inventory system. The sales force's compensation package affects how the salespeople are going to exert their effort, which in turn determines the sales pattern for the firm's product and ultimately drives the performance of the firm's production and inventory system. In general, a smooth demand process facilitates production/inventory planning. Therefore, it is beneficial for a firm to induce its salespeople to exert effort in a way that actually smoothes the demand process. The article proposes a compensation package to induce such behavior. It evaluates and compensates the sales force on a moving-time-window basis, where the length of the time window is determined by the production lead time. Numerical examples show that the proposed package is beneficial to the firm relative to a widely used compensation plan based on annual quotas.
(*Sales-Force Compensation; Agency Theory; Sales-Force Management; Demand Smoothing; Production/Inventory Planning*)

1. Introduction

There is a great deal of research on the problem of sales-force compensation. The firm benefits from selling effort, but effort is costly to the sales force, resulting in a misalignment of objectives between the firm and its salespeople. Thus a reward system is needed to correct this misalignment. The existing research provides guidelines that can help a firm structure an appropriate reward system.

So far, the theoretical papers in the area of sales-force management have largely ignored the impact of sales-force behavior as induced by a given reward system on the firm's production and inventory system. If salespeople are evaluated on an annual basis according to a quota-based plan, it is not surprising to see that sales magically surge at the end of the year, exhibiting the so-called "hockey stick" phenomenon.[1] This sales pattern causes difficulties in production planning and thus increases the firm's operational costs.

The objective of this paper is to investigate sales-force compensation schemes in light of their impact on the firm's production/inventory system. It is well known from the operations management literature that an inventory system with a smoother demand process, i.e., less variability, incurs less overage and

[1]There are other possible explanations for the hockey stick phenomenon. For example, buyers may have more incentive to buy as the year comes to an end, since they face less budget uncertainty. It may even have something to do with the human psyche (e.g., a year as a natural unit of measurement of time) and various societal rhythms (e.g., holidays). Moreover, the phenomenon also arises in forms other than the flow of physical goods. In financial markets, it has often been observed that stocks that have been doing well (resp., poorly) in a quarter tend to do even better (resp., worse) toward the end of the quarter. One explanation is that at the end of a quarter, mutual fund managers tend to sell laggards and buy winners in order to boost their quarterly performance. This phenomenon is known as "window dressing." The author thanks Mark Broadie for a discussion of this phenomenon.

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underage costs. Therefore, the firm benefits if it can induce its salespeople to exert selling effort in a way that actually smoothes the demand process. As it turns out, this is possible, and the resulting benefits to the firm can be substantial.

We consider a model where a firm sells a single product through a single-agent sales force. The demand in a period is jointly determined by the selling effort exerted in the period and a random shock. The firm cannot directly observe the selling effort and thus can only reward the salesperson based on the realized demand. This article begins with a prevalent compensation package, a quota-based plan, that rewards the sales force on an annual basis: The salesperson earns a salary and a commission income that is a prespecified fraction of the annual sales in excess of the quota. Widely used in practice, this annual-quota (AQ) plan has also been justified on theoretical grounds; see, e.g., Basu et al. (1985) and Raju and Srinivasan (1996). We show how the firm can determine the optimal contract parameters for the AQ plan after taking into account its impact on sales as well as production and inventory costs.

To understand how the AQ plan can be improved, the article proceeds to consider the first-best (FB) scenario, where it is assumed that selling effort is costlessly observable and legally contractible. In this case, the firm can instruct the sales force to follow a given effort strategy, and the problem reduces to identifying an integrated effort/replenishment strategy to maximize the firm's profits. The FB solution suggests an effort strategy that in fact smoothes the demand process. Based on the structure of the FB solution, we propose an alternative compensation package to induce the demand-smoothing behavior from the sales force (when effort is not observable).

The proposal is the so-called moving-time-window (MW) plan. At the end of each period t (e.g., month), the firm determines the total sales in the current period as well as the past L periods, where L is the production lead time. Let the total be w_t. The salesperson earns a fixed bonus only if w_t reaches a predetermined quota. Therefore, it is still a quota-based plan, but sales performance is evaluated every period based on the most recent lead-time demand (i.e., w_t). Numerical examples show that, compared with the AQ plan, the MW plan can sometimes substantially increase the firm's profits.

This article contributes to the existing sales-force management literature in the following way. In designing a sales-force compensation package, there are invariably two basic questions: 1) How often should the salespeople be evaluated (e.g., monthly, quarterly, or annually)? and 2) At the time of an evaluation, how should sales performance be measured, for example, under monthly evaluations, should the performance be based on sales in the most recent month or the most recent quarter? It appears that the existing literature does not distinguish between these two questions; e.g., a monthly plan evaluates salespeople at the end of each month based on sales in the current month. Therefore, sales performance is evaluated in nonoverlapping time windows. So how long should the performance window be? In a survey of 200 compensation plans, 31% paid salespeople incentive earnings on an annual basis, 8% semiannually, 28% quarterly, and 33% monthly (Churchill et al. 1993, p. 590). It is unclear on what basis these performance windows were chosen. Churchill et al. offer one rationale: Shorter windows increase the motivating power of the plan but add to the administrative expenses, suggesting that quarterly plans appear to be a good compromise. This article adds another dimension to this trade-off: the benefit of demand smoothing. To induce the sales force to smooth the demand process, its performance should be evaluated as often as possible (every period in the model),[2] with the length of the performance window determined by the production lead time (which can be multiple periods). Therefore, the performance windows associated with different evaluations often overlap.

The literature on sales-force management is voluminous. It roughly divides into two groups. One assumes deterministic sales-response functions (of selling effort); the other stochastic. The first group considers such issues as how commission rates can be optimally set in a multiproduct environment with a

[2]The optimal frequency of performance evaluation should also take into account the cost in administering a compensation plan as well as its psychological impact on the sales force (a high evaluation frequency may decrease the potential incentive earnings at each review to such a small amount that it no longer has any motivating power).

multiperson sales force, when the pricing decision should or should not be delegated to the sales force, and the impact of dynamic effort decisions on sales-force compensation. Sample references are Farley (1964), Davis and Farley (1971), Tapiero and Farley (1975), Weinberg (1975, 1978), and Srinivasan (1981). The literature assuming a stochastic demand function is built upon agency theory in economics. The seminal papers in agency theory include Harris and Raviv (1978, 1979), Holmstrom (1979, 1982), Shavell (1979), and Grossman and Hart (1983). Basu et al. (1985) are the first to apply the agency theory to the sales-force problem, characterizing the form of an optimal compensation plan in a single-product, single-agent setting. This model is then extended by a stream of research to allow for multiple sales agents, multiple products, asymmetric information about sales-force productivity, delegation of pricing decisions, and so on. For more details, see Lal (1986), Lal and Staelin (1986), Dearden and Lilien (1990), Rao (1990), Lal and Srinivasan (1993), and Raju and Srinivasan (1996). Porteus and Whang (1991) further extend this literature by connecting sales-force incentives to manufacturing incentives. Their model is single-period and thus ignores the dynamic nature of sales-force behavior. Comprehensive reviews are provided by Baiman (1982) for agency theory and by Coughlan and Sen (1990) and Coughlan (1993) for the sales-force literature.

The rest of this article is organized as follows. Section 2 presents the model basics. Section 3 analyzes the annual-quota plan. Section 4 characterizes the first-best solution. Section 5 describes the moving-time-window plan. Section 6 contains the numerical examples. Section 7 concludes.

2. Model Basics

A firm sells a single product through a single agent. The selling price of the product is fixed; the agent is not empowered to give price discounts to customers. The only way to increase sales is for the agent to expend selling effort (e.g., making sales calls and visiting customer sites). Let D_t be the demand in period t, which is assumed to be

$$D_t = \xi_t + e_t,$$

where ξ_t is the random shock and e_t the selling effort in period t. Moreover, we assume that $\xi_1, \xi_2, \cdots$ are independent and identically distributed. Both ξ_t and e_t take only nonnegative values. The above sales-response function has appeared in Lal and Staelin (1986), Rao (1990), and Lal and Srinivasan (1993).

The sales agent's utility is a function of income and effort. We assume that the agent assesses her utility on an annual basis.[3] Let $H(w, e)$ be the agent's utility in a year if her annual income is w and annual effort is e. Assume

$$H(w, e) = U(w) - V(e),$$

with $U(\cdot)$ strictly concave, increasing, and twice differentiable, and $V(\cdot)$ convex, increasing, and twice differentiable. Therefore, the agent is risk averse, and she dislikes exerting effort (all else being equal). This additive utility function has been used by others; see, e.g., Holmstrom (1979).

The sales agent decides how much effort to exert in each period. The objective is to maximize her expected utility.

The firm keeps a finished-goods inventory, and a production manager (PM) is responsible for replenishing it. In each period, he decides how much to produce. The production lead time is constant. When the firm runs out of finished-goods inventory, the customer demand is assumed to be backlogged. The firm incurs variable costs for production and procurement, as well as linear inventory holding costs and linear back-order penalty costs. The PM's objective is to minimize the long-run average operational costs, i.e., variable, holding, and back-order costs, given the demand process that is shaped by the sales agent's effort decisions.

The firm's objective is to maximize its long-run average profits, i.e., revenues minus sales-force compensation minus operational costs, subject to the constraint that the expected utility for the sales agent is at least U_0, the agent's reservation utility. This constraint is required to keep the agent from leaving the firm. Because the selling price is fixed and the agent controls the selling effort, the long-run average revenues are beyond the control of the PM. Thus the PM's objective is consistent with the firm's, a simplifying assumption. This

[3] For convenience, the sales agent is "she." Later, the production manager is "he."

is, however, not true for the agent since she dislikes exerting effort. Thus the firm must motivate the agent to work (through a compensation plan). The rest of this paper examines various types of compensation plans.

We close this section with a summary of basic notation:

p = unit selling price.
D_t = demand in period t.
ξ_t = random shock in period t.
$f(\cdot)$ = probability density function of ξ_t, $t = 1, 2, \cdots$.
$F(\cdot)$ = cumulative distribution function of ξ_t, $t = 1, 2, \cdots$.
e_t = selling effort in period t.
K = number of periods in a year.
c = variable production/procurement cost.
h = holding cost rate.
b = back-order cost rate.
L = replenishment lead time, a nonnegative integer.
$H(w, e)$ = sales agent's utility function, w annual income, and e annual effort
= $U(w) - V(e)$, $U(\cdot)$ strictly concave, increasing, and twice differentiable, and $V(\cdot)$ convex, increasing, and twice differentiable.

3. The Annual-Quota System

This section considers a prevalent reward system, which compensates the sales agent at the end of each year according to a quota-based plan. It has a quota q and a commission rate β. If the annual demand[4] x (in physical units) exceeds the quota, the agent makes commissions on the excess at rate β. That is, $\beta(x - q)^+$ equals total commissions for the year. In addition, the agent receives an annual salary, α (≥ 0). The sequence of events is: 1) the firm specifies a contract (i.e., α, q, and β); 2) the agent makes effort decisions under the contract; and 3) the PM makes production decisions. The contract, once determined, does not change over time. The firm chooses the contract parameters to maximize its long-run average profits. We begin with the agent's response to a given contract.

[4]We use demands and sales interchangeably in this paper.

3.1. The Agent's Response

Recall that the sales agent's objective is to maximize her expected utility. Because the utility is calculated on an annual basis, the planning horizon for the agent is effectively one year. Moreover, because neither the compensation package nor the characteristics of the random shocks change over time, the problem facing the agent is the same every year. Suppose period 1 is the first period of a year. Because there are K periods in a year, the objective is to maximize

$$E\left[U\left(\alpha + \beta\left(\sum_{t=1}^{K} D_t - q\right)^+\right) - V\left(\sum_{t=1}^{K} e_t\right)\right].$$

Note that there exists an optimal solution with $e_1 = \cdots = e_{K-1} = 0$, i.e., the selling effort, if any, is concentrated in the last period of the year. This is because the value of the objective function is not affected if e_t for some $t < K$, which may be a function of the sales in the periods before t, is postponed to period K. Therefore, the hockey stick phenomenon arises in this model. At the beginning of period K, before e_K is decided, the agent has observed the values of $\xi_1, \xi_2, \cdots, \xi_{K-1}$. Let $\hat{\xi}_K$ be the sum of these observed values. For convenience, replace e_K with e. The agent's problem becomes

$$\max_e E_{\xi_K}\left[U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+) - V(e)\right].$$

Of course, the optimal solution is a function of $\hat{\xi}_K$. Let it be $e(\hat{\xi}_K)$.

It is quite intuitive why the agent wants to postpone the effort decision to the last period of the year. At any point in time, the marginal benefit of effort depends on past as well as future sales during the year. Postponing the effort decision allows the agent to learn more about the marginal benefit of effort. Because the cost of effort (i.e., the value of $V(\cdot)$) is unaffected by the allocation of effort over time as long as the annual total remains fixed, it is not surprising to see that all effort is concentrated in the last period.[5]

[5]In reality, it is possible to see selling effort exerted in other periods as well. Perhaps this is because the amount of effort that can be exerted in a period is often limited, e.g., there are only 24 hours in a day. Our model assumes no such limits. This is a reasonable simplification if the period in the model is fairly long, such as a quarter or a month.

The agent's objective function is generally not well behaved. The following lemma provides an easy-to-compute upper bound that is useful in solving the problem numerically. (All omitted proofs can be found in the Appendix, unless otherwise mentioned.)

LEMMA 1. *$e(\hat{\xi}_K) \le \hat{e}$ for any $\hat{\xi}_K$, where*

$$\hat{e} = argmax_e$$
$$E_{\xi_K}[U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)) - V(e)].$$

The following lemma is expected.

LEMMA 2. *$\partial e(\hat{\xi}_K)/\partial\alpha \le 0$ for any $\hat{\xi}_K$, i.e., increasing salary reduces the agent's incentive to sell.*

3.2. The PM's Problem

Suppose the PM knows the sales agent's decision rule. Thus, he knows that the selling effort in a year is concentrated in the last period, the amount of which is $e(\hat{\xi}_K)$, a function of the cumulative demand in that year before the last period. This periodic effort stream merges with the random stream $\{\xi_t\}$ to form the demand process for the firm's product. Given this demand process, the PM determines a replenishment strategy to minimize the long-run average operational costs, i.e., variable costs and holding and back-order costs. Because all demands are eventually satisfied (due to backlogging), the long-run average variable costs are fixed. We thus focus on holding and back-order costs.

Define (finished-goods) *inventory position* to be outstanding orders plus on-hand inventory minus back orders, and *inventory level* to be on-hand inventory minus back orders. Let $IP(t)$ be the inventory position at the beginning of period t after ordering and $IL(t)$ the inventory level at the end of period t. (We follow the convention of charging holding and back-order costs based on period-ending inventory levels.)

A period is of type k if it is the kth period in a year, $k = 1, \cdots, K$. Let k_t be the type of period t. Let $\hat{D}_t$ be the total sales in periods $t - 1, t - 2, \cdots, t - k_t + 1$, with $\hat{D}_t = 0$ if $k_t = 1$. Since period $t - k_t + 1$ is the first period in the year that contains period t, $\hat{D}_t$ is the total sales before period t in that year. Since effort is exerted only in type-K periods,

$$\hat{D}_t = \xi_{t-1} + \xi_{t-2} + \cdots + \xi_{t-k_t+1}$$

for all t with $k_t > 1$. Define $S_t = (k_t, \hat{D}_t)$ to be the state of the inventory system at the beginning of period t. Note that S_t completely determines the probabilistic characteristics of the demand process after t. For example, if $S_t = (k, z)$ for some $k < K$, then $D_t = \xi_t$; if $S_t = (K, z)$, then $D_t = \xi_t + e(z)$. It is thus conceivable that the optimal replenishment policy in period t depends on S_t.

The stochastic process $\{S_t\}$ is a Markov chain, with one-step transition probabilities

$$\Pr(S_{t+1} = (1,0) \mid S_t = (K, z)) = 1,$$

and

$$\Pr(S_{t+1} = (k + 1, z_{t+1}) \mid S_t = (k, z_t)) =$$
$$\Pr(\xi_t = z_{t+1} - z_t),\ k = 1, \cdots, K - 1,$$

and $\Pr(S_{t+1} \mid S_t) = 0$ otherwise. Let S be the state space of the Markov chain.

The PM's problem is somewhat similar to the inventory problem studied in Chen and Song (1997), where the demand process is driven by an exogenous Markov chain (whose evolution is independent of the operations of the inventory system). In our model, however, the Markov chain $\{S_t\}$ driving the demand process is no longer exogenous. As it turns out, one can still use the Chen-Song approach to characterize the optimal policy. We state the following result without proof, which can be found in Chen (1998).

THEOREM 1. *The optimal replenishment policy is to follow a base-stock policy with an order-up-to level $y^*(s)$ when the state of the Markov chain is s, $s \in S$. That is, for any period t with $S_t = s$, if the inventory position is below $y^*(s)$, order to increase the inventory position to $y^*(s)$; otherwise, do not order.*

The optimality proof in Chen (1998) is in fact an algorithm for finding the optimal state-dependent base-stock levels. However, the algorithm in its current form is rather complex. Below, we first show how an optimal solution can easily be obtained in a special case, and then present a dynamic programming algorithm that generates a heuristic, and sometimes optimal, solution.

Let $D[t_1, t_2]$ be the total demand in periods $t_1, \cdots, t_2$. The following is the well-known inventory balance equation:

$$IL(t + L) = IP(t) - D[t, t + L].$$

Given $IP(t) = y$ and $S_t = s \in S$, the expected holding and back-order costs in period $t + L$ are

$$G(y \mid s) = E[h(y - D[t, t + L])^+ + b(y - D[t, t + L])^- \mid S_t = s]. \quad (1)$$

We charge $G(y \mid s)$ to period t if $IP(t) = y$ and $S_t = s \in S$. Clearly, $G(y \mid s)$ is convex in y and is minimized at a finite point, which we denote by $y = y^o(s)$. Thus $y^o(s)$ is the *myopic base-stock level* for any period with state s, i.e., setting the inventory position at this level minimizes the expected holding and back-order costs one lead time later. For convenience, we also write $y^o(k, z)$ for $y^o(s)$ if $s = (k, z)$.

THEOREM 2. *If $L = 0$, then the myopic base-stock policy is optimal. That is, for any period t with $S_t = s$, if the inventory position is below $y^o(s)$, order to increase the inventory position to $y^o(s)$; otherwise, do not order.*

Now suppose $L > 0$. Below, we present a dynamic programming (DP) algorithm for finding a heuristic solution. The DP algorithm relaxes the linkage between consecutive years, reducing the infinite planning horizon to a single year.

Let period 1 be the first period of a year. Suppose the planning horizon is only one year, i.e., we try to minimize the expected holding and back-order costs in periods $1, 2, \cdots, K$. Recall that the costs charged to a period are $G(y \mid s)$ if the state is s and the inventory position after ordering is y. Let $H_k(w, z)$ be the minimum expected holding and back-order costs in periods $k, k + 1, \cdots, K$ given that the inventory position *before* ordering in period k is w and the state is $(k, z) \in S$, $k = 1, \cdots, K$. This function can be computed recursively. First, consider period K, the last period in the planning horizon. Because $G(y \mid (K, z))$ is convex in y and is minimized at $y = y^o(K, z)$,

$$H_K(w, z) = G(\max\{w, y^o(K, z)\} \mid (K, z)). \quad (2)$$

That is, if $w < y^o(K, z)$, order to increase the inventory position to $y^o(K, z)$; otherwise, do not order. The following is the dynamic program recursion:

$$H_k(w, z) = \min_{y \geq w} [G(y \mid (k, z)) + EH_{k+1}(y - \xi_k, z + \xi_k)], \quad k = 1, \cdots, K - 1, \quad (3)$$

where y is the inventory position after ordering in period k. Equation (3) uses the fact that no effort is exerted in period k, and thus $D_k = \xi_k$ for $k = 1, \cdots, K - 1$.

LEMMA 3. *$H_k(w, z)$ is convex in w for all $(k, z) \in S$, $k = 1, \cdots, K$.*

Therefore, solving the dynamic program amounts to minimizing a sequence of convex functions. Let $y^d(k, z)$ be a minimum point of $G(y \mid (k, z)) + EH_{k+1}(y - \xi_k, z + \xi_k)$, $k = 1, \cdots, K - 1$, and $y^d(K, z) = y^0(K, z)$. The optimal DP solution is a state-dependent base-stock policy: If the state is (k, z), place an order to increase the inventory position to $y^d(k, z)$, and do not order if the inventory position before ordering is already above the target. The minimum expected total cost over the one-year planning horizon is

$$H_1(y^d(1, 0), 0). \quad (4)$$

The DP solution is certainly feasible for the original problem (with an infinite planning horizon). The resulting long-run average holding and back-order costs per year are, however, likely to be different from Equation (4), which is a lower bound on the long-run average cost of any feasible policy.

The DP solution is optimal when it has the so-called *maximum property* (Zipkin 1989), i.e., the inventory position before ordering at the beginning of each year does not exceed $y^d(1, 0)$. This holds if and only if

$$y^d(k, z) - (D[k, K] \mid (k, z)) \leq y^d(1, 0), \quad \forall (k, z) \in S, \quad (5)$$

where $(D[k, K] \mid (k, z))$ denotes the total demand in periods $k, \cdots, K$ given $S_k = (k, z)$. This turns out to be true when $K = 2$.

THEOREM 3. *The DP solution is optimal if $K = 2$.*

3.3. Choosing Contract Parameters

After characterizing the sales agent's effort decisions and the production manager's replenishment decisions, we now turn to the problem of choosing the contract parameters α, q, and β. This is a typical principal-agent problem with the firm as the principal and the sales force as the agent.

Let $\Pi(\alpha, q, \beta)$ be the firm's long-run average profits per year. The problem facing the firm is to maximize

$\Pi(\alpha, q, \beta)$ in anticipation of the responses of the sales agent and the production manager. (Thus the firm is risk neutral.) Recall that the sales agent's optimal response is to exert all the selling effort in a year in the last period, and the effort level is a function of the total demand in the prior $K - 1$ periods. This function is

$$e(\hat{\xi}_K) = \text{argmax}_e \; E_{\xi_K}[U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+) - V(e)] \quad (6)$$

where $\hat{\xi}_K$ is, again, the demand in the first $K - 1$ periods of a year. Thus the annual demand is $\hat{\xi}_K + \xi_K + e(\hat{\xi}_K)$, and the expected gross profits (revenues $-$ variable costs) per year are $(p - c)E[\hat{\xi}_K + \xi_K + e(\hat{\xi}_K)]$. Let $c_{hb}(\alpha, q, \beta)$ be the minimum long-run average holding and back-order costs per year. Therefore,

$$\Pi(\alpha, q, \beta) = (p - c)E[\hat{\xi}_K + \xi_K + e(\hat{\xi}_K)] - c_{hb}(\alpha, q, \beta) - \alpha - \beta E[\hat{\xi}_K + \xi_K + e(\hat{\xi}_K) - q]^+$$

where the last two terms represent the total payment to the sales force (salary plus commissions). To keep the agent from leaving the firm, the firm must ensure that her expected payoff is not lower than U_0, i.e.,

$$EU(\alpha + \beta(\hat{\xi}_K + \xi_K + e(\hat{\xi}_K) - q)^+) - E[V(e(\hat{\xi}_K))] \geq U_0. \quad (7)$$

This is often referred to as the *participation constraint* in agency theory. The firm's problem can thus be written as

$$\max_{\alpha,q,\beta} \Pi(\alpha, q, \beta) \quad \text{s.t. Constraints (6) and (7).} \quad (8)$$

We evaluate the objective function by simulation. The optimal solution can be obtained via a search. Let (α^*, q^*, β^*) be an optimal solution.

The following heuristic bounds are useful in obtaining the optimal contract parameters. First, we conjecture that if $\alpha^* > 0$, then the participation constraint must be binding.[6] As a result, $U(\alpha^*) \leq U_0$ if $\alpha^* > 0$. (Suppose $U(\alpha^*) > U_0$. In this case, if the agent chooses to exert zero effort, her expected utility will be at least $U(\alpha^*)$, which already exceeds her reservation utility. Therefore, under the agent's optimal response, the participation constraint is not binding—a contradiction.) Let $\hat{\alpha}$ be the solution to $U(\hat{\alpha}) = U_0$. Then $0 \leq \alpha^* \leq \hat{\alpha}$. Second, it is reasonable that $0 \leq \beta^* \leq p - c$ since the firm's profit margin does not exceed $p - c$. Finally, as to the quota, note that it serves to limit the commission income due to the random shocks, which have nothing to do with selling effort. If q is larger than $K\bar{\xi}$, where $\bar{\xi}$ is an upper bound on the random shock in a period (assuming it is bounded), then the commission income, if any, comes from sales effort. Consequently, we restrict to $0 \leq q^* \leq K\bar{\xi}$.

[6]The intuition is that increasing salary decreases the sales agent's incentive to sell (Lemma 2), and thus decreases the firm's revenue. Also, a higher salary leads to a higher expected payment to the agent. Consequently, a positive salary makes sense only when the participation constraint is binding.

4. The First-Best Solution

In this section, we assume that selling effort is costlessly observable and legally contractible. In this case, the firm can instruct the sales agent to follow any given effort strategy and pay her a salary that makes the participation constraint binding. If the agent does not follow the instruction, the firm can make her compensation so low that this becomes a less attractive option. In other words, the firm can impose a forcing contract on the agent. Let $\tilde{e}$ be the total effort in a year, which may be a random variable. From the binding participation constraint, we have the agent's salary $\alpha = U^{-1}(U_0 + EV(\tilde{e}))$, where $U^{-1}(\cdot)$ is the inverse function of $U(\cdot)$. (Because the firm is risk neutral and the agent is risk averse, a salary is the firm's optimal way of compensating the agent.) The firm's objective is to choose an integrated effort/replenishment strategy to maximize its long-run average profits (i.e., revenues minus operational costs minus agent's salary). The result is the first-best solution.

The first-best solution provides a benchmark for the original scenario with unobservable selling effort. More important, as we will see later, it leads to an alternative reward system that can improve the firm's profits. In general, however, the problem facing the firm is complex; we begin with simpler special cases to build intuition.

4.1. Zero Lead Time

When the replenishment lead time is zero, i.e., $L = 0$, the first-best solution is easy to obtain. Suppose we are at the beginning of period t, and it has been decided that the selling effort for the period is going to be e_t. Thus $D_t = \xi_t + e_t$. Consider the replenishment decision. If $IP(t) = y$, the expected holding and back-order costs in period t are equal to $E[h(y - \xi_t - e_t)^+ + b(y - \xi_t - e_t)^-]$, a convex function of y minimized at $y = y_0 + e_t$, where y_0 is a minimum point of $g(y) = E[h(y-\xi_t)^+ + b(y - \xi_t)^-]$. Suppose $IP(t) = y_0 + e_t$, the myopic base-stock level minimizing the one-period costs. At the end of period t, the inventory position is $y_0 + e_t - D_t \le y_0$. Thus we can place an order to ensure $IP(t + 1) = y_0 + e_{t+1}$, the myopic base-stock level for the next period, and so on. In other words, the minimum expected holding and back-order costs per period, i.e., $g(y_0)$, can be achieved regardless of the effort strategy. Now let e be the total selling effort in a year. The firm's annual profits can be written as a constant plus $(p - c)e - U^{-1}(U_0 + V(e))$, a concave function of e. (Because $V(\cdot)$ is convex, it is optimal for the firm to keep selling effort constant for every year.) Let this function be maximized at $e = e^*$.

Theorem 4. *If $L = 0$, then the optimal replenishment strategy is to order up to $y_0 + e_t$ in period t, where e_t is the selling effort in the period, and the optimal effort strategy is to have the total selling effort in every year equal to e^*. It is not important how the effort is allocated across time periods.*

4.2. Linear Disutility of Effort

Now suppose the replenishment lead time is positive. In this case, the first-best solution becomes very complex. This is because the optimal effort decisions must take into account the entire vector of incoming orders so as to better match supply with demand.[7] To simplify, we assume that $V(x) = ax$ for some positive constant a.

When the effort and replenishment decisions are integrated, it becomes apparent that the sales force has the so-called *demand-smoothing function*. Consider a simple example. Suppose $L = 1$ and the random shock ξ_t has a finite support [0, 5]. Let the effort strategy be $e_t = 5 - \xi_{t-1}$. What will the minimum long-run average holding and back-order costs be? Take any period t. Note that

$$D[t, t + L] = D_t + D_{t+1} = \xi_t + e_t + \xi_{t+1} + e_{t+1} = e_t + 5 + \xi_{t+1}.$$

Thus the myopic policy for period t, i.e., one that minimizes the expected holding and back-order costs one lead time later (in period $t + 1$), is to order up to $y_0 + e_t + 5$, and the resulting minimum one-period cost is $g(y_0)$. (Recall that y_0 is a minimum point of $g(y) = E[h(y - \xi_t)^+ + b(y - \xi_t)^-]$.) Assume $IP(t) = y_0 + e_t + 5$. Thus the inventory position at the end of period t is $IP(t) - D_t = y_0 + 5 - \xi_t \le y_0 + e_{t+1} + 5$, the myopic base-stock level for period $t + 1$. This indicates that the myopic base-stock level can be reached in every period, and the minimum long-run average holding and back-order costs are $g(y_0)$ per period. Note that this is also the minimum holding and back-order costs when $L = 0$. With no selling effort, the variance of the lead-time demand would be $2\text{Var}[\xi_t]$; the above effort strategy reduces the variance to $\text{Var}[\xi_t]$. Therefore, sales effort, when properly exerted, can smooth the demand process, reducing holding and back-order costs.

Interestingly, the form of the above effort strategy is myopically optimal (among all feasible policies). Suppose we are at the beginning of period t, having observed $IP(t) = y$. The expected holding and back-order costs incurred one lead time later are

$$\begin{aligned} G(y) &\overset{\text{def}}{=} E[h(y - D[t, t + L])^+ + b(y - D[t, t + L])^-] \\ &= (h + b)\int_0^y \Pr(D[t, t + L] \le x)dx - by + bED[t, t + L]. \end{aligned} \tag{9}$$

Suppose we want to minimize $G(y)$ through exerting sales effort in periods $t, t + 1, \cdots, t + L$. Without loss of generality, we restrict to strategies that exert all the effort (in periods $t, t + 1, \cdots, t + L$) in the last period of the interval. (This is the postponement idea used in § 3.1.) Let θ be the effort level in period $t + L$, which may be a function of $D_t, D_{t+1}, \cdots, D_{t+L-1}$ but not ξ_{t+L}

[7] To minimize underage and overage costs, it is desirable to have a high level of effort in a period with high expected supply and to have a low level of effort when shortage is likely. Therefore, it is not true that the firm prefers higher effort every period. This is in contrast with a typical marketing model where more selling effort always increases the firm's profits (before compensating the sales force).

since the selling effort in period $t + L$ must be decided before observing ξ_{t+L}. (Note that D_τ, $\tau = t, \cdots, t + L - 1$, may be different from ξ_τ because of earlier effort decisions.) Thus

$$D[t, t + L] = D_t + D_{t+1} + \cdots + D_{t+L-1} + \xi_{t+L} + \theta(D_t, D_{t+1}, \cdots, D_{t+L-1}).$$

Clearly, we can restrict θ to be a function of $\hat{D} \stackrel{\text{def}}{=} \Sigma_{\tau=t}^{t+L-1} D_\tau$ only. As a result,

$$D[t, t + L] = \hat{D} + \theta(\hat{D}) + \xi_{t+L}. \quad (10)$$

If $\theta(\cdot)$ minimizes $G(y)$ for all y while keeping $E(\theta(\hat{D}))$ constant, then it is called a *myopic effort strategy*.[8] The following theorem establishes the form of such a strategy.

THEOREM 5. *Let X be a nonnegative random variable. For any function $\theta(\cdot) \geq 0$ and any nonnegative real number B with $E(B - X)^+ = E\theta(X)$,*

(i) $\int_0^z Pr(X + (B - X)^+ \leq x)dx \leq \int_0^z Pr(X + \theta(X) \leq x)dx$, $\forall\, z \geq 0$, and

(ii) $Var[X + (B - X)^+] \leq Var[X + \theta(X)]$ with the former nonincreasing in B.

Recall that a myopic effort strategy $\theta(\cdot)$ minimizes $G(y)$ for all y while keeping $E(\theta(\hat{D}))$ constant. That is, from Equations (9) and (10), $\theta(\cdot)$ minimizes $\int_0^y \Pr(D[t, t + L] \leq x)dx$ for all y. From Equation (10) and the fact that $\hat{D}$ is independent of ξ_{t+L},

$$\int_0^y \Pr(D[t, t + L] \leq x)dx = \int_{x=0}^{y} \int_{z=0}^{x} \Pr(\hat{D} + \theta(\hat{D}) \leq x - z)dF(z)dx$$
$$= \int_{z=0}^{y} dF(z) \int_{x=z}^{y} \Pr(\hat{D} + \theta(\hat{D}) \leq x - z)dx$$
$$= \int_{z=0}^{y} dF(z) \int_{x'=0}^{y-z} \Pr(\hat{D} + \theta(\hat{D}) \leq x')dx'.$$

Therefore, it suffices for $\theta(\cdot)$ to minimize $\int_{x'=0}^{y-z} \Pr(\hat{D} + \theta(\hat{D}) \leq x')dx'$ for all y and $z \leq y$. From Theorem 5(i), this is achieved with $\theta(x) = (B - x)^+$ for some nonnegative real number B. From Theorem 5(ii), this myopic effort strategy also minimizes the variance of the lead-time demand $D[t, t + L]$ while keeping its mean constant.

Based on the above analysis, we suggest the following heuristic first-best solution. Let the sales effort in period t be

$$e_t = (B - D_{t-1} - \cdots - D_{t-L})^+ \quad (11)$$

for some nonnegative constant B. On the other hand, replenishment follows a base-stock policy with order-up-to level $Y + e_t$, where Y is a constant. As in the zero lead-time case, one can show that

$$IP(t) = Y + e_t, \quad \forall t. \quad (12)$$

The heuristic solution is parameterized by B and Y. In general, the above effort/replenishment strategy leads to a complex demand/supply process. We use simulation for evaluating the firm's long-run average profits and search for the optimal B and Y.

4.3. Convex Disutility of Effort

When $V(\cdot)$ is strictly convex, it is desirable to reduce the uncertainty in the annual sales effort. We thus change the effort strategy to

$$e_t = A + (B - D_{t-1} - \cdots - D_{t-L})^+ \quad (13)$$

for some nonnegative constants A and B. If $B = 0$, the sales effort stays constant from period to period and thus from year to year. The replenishment strategy remains the same as in Equation (12). Now there are three strategy parameters (A, B, Y) whose optimal values can again be obtained via a search using simulation.

5. The Moving-Time-Window Plan

We now return to the original scenario where the firm cannot directly observe the selling effort and thus must rely on the realized demands for rewarding the sales agent. The annual-quota system considered earlier does not take into account the impact of selling effort on the firm's production and inventory system, whereas the first-best solution suggests that sales effort, when properly allocated over time, can smooth the demand process and reduce operational costs. This

[8]If the expected effort remains constant in every period, the firm's long-run average revenues and variable costs are fixed. Moreover, because the expected annual effort is also constant, the sales agent's salary remains fixed because the disutility of effort is linear as assumed earlier. We can, therefore, focus on the holding and back-order costs.

section provides a plan that induces this demand-smoothing behavior.

Consider the following reward system. At the end of each period, t, the firm determines the total demand in the past $L + 1$ periods, w_t, i.e.,

$$w_t = D_t + D_{t-1} + \cdots + D_{t-L}.$$

The sales agent earns a fixed bonus γ in period t only if w_t is greater than or equal to m, a predetermined quota. In addition to the bonuses (if any), the firm pays the agent an annual salary, which is again denoted by α. This compensation package will be referred to as the moving-time-window system. The firm chooses the contract parameters (α, m, γ) to maximize its long-run average profits.

We now develop a plausible response by the sales agent to the above reward system. Suppose the agent is now at the beginning of period t, trying to determine e_t. Although this decision affects her bonuses in periods $t, t + 1, \cdots, t + L$, the only reason for the agent to choose a positive effort level, i.e., $e_t > 0$, is to earn a bonus in period t. (Sales effort can always be postponed if the objective is to make a bonus in a future period.) Let u_t be the total demand in the past L periods, i.e., $u_t = D_{t-1} + D_{t-2} + \cdots + D_{t-L}$. Thus

$$w_t = u_t + e_t + \xi_t.$$

If $e_t > 0$, it should be just large enough to provide a sufficient probability for a bonus in period t, i.e., increasing $\Pr(u_t + e_t + \xi_t \geq m)$ to a threshold level. On the other hand, the agent may choose $e_t = 0$. This arises either because the random shock ξ_t and the past demands u_t already provide a sufficient probability for a bonus in period t, or because u_t is so low that the effort required to earn a bonus in period t is too great. The above discussion suggests the following effort strategy: $e_t > 0$ if and only if $\underline{u} \leq u_t < \bar{u}$, in which case $e_t = \bar{u} - u_t$, where $\underline{u}$ and $\bar{u}$ are policy parameters chosen by the agent to maximize her expected utility.

Remark. If the sales agent is risk neutral and the disutility function is linear, i.e., $U(w) = w$ and $V(x) = ax$ for some positive constant a, then the above effort strategy is myopically optimal under rather mild conditions. To see this, note that the expected bonus in period t is $\gamma\Pr(\xi_t \geq m - e - u)$. Because the disutility of effort is ae, the expected utility for the period is

$$\gamma\Pr(\xi_t \geq m - e - u) - ae$$
$$= \gamma(1 - F(m - e - u)) - ae.$$

Maximizing the above expected utility is equivalent to

$$\min_{y \geq u} z(y) \overset{\text{def}}{=} \gamma F(m - y) + ay$$

where $y = u + e$. It is easy to see that the above ($\underline{u}$, $\bar{u}$) policy solves this optimization problem if, e.g., either $z(\cdot)$ or $-z(\cdot)$ is unimodal in $(0, m)$. The necessary conditions are much more general than these unimodality conditions.

We assume that the production manager knows the agent's effort strategy and, based on that, determines a replenishment policy to minimize the firm's holding and back-order costs. (Note, again, that the firm's revenue, variable cost, and payment to the sales agent are not affected by the PM's decisions.) As in the first-best solution, we assume that the PM follows a base-stock policy with order-up-to level $Y + e_t$ in period t for some constant Y.

Finally, anticipating the responses of the sales agent and the production manager, the firm chooses the contract parameters (α, m, γ) to maximize its long-run average profits, subject to the participation constraint. We again rely on simulation for finding the optimal contract parameters.

6. Numerical Examples

In this section, we use numerical examples to compare the following three scenarios: the annual-quota system (AQ), the first-best scenario (FB), and the moving-time-window system (MW). AQ represents the status quo, MW is an alternative, and FB is the ultimate, albeit unrealistic, goal.

We first specify the numerical examples. There are 12 periods (months) in a year, i.e., $K = 12$. The random shocks have a binomial distribution with parameters 10 and 0.5, i.e.,

$$f(x) = \binom{10}{x} 0.5^{10}, \quad x = 0, 1, \cdots, 10.$$

The sales agent's utility function is $H(w, e) = 5\sqrt{w} - 0.1e^2$. The selling price is $p = 15$ per unit. The variable cost is $c = 12$ per unit. The back-order penalty cost is

$b = 10$ per unit per period. We vary the remaining parameters to yield eight examples; see Table 1. For each example, we solved the principal-agent problems associated with AQ and MW and the centralized decision-making problem under FB.

Figure 1 depicts the firm's profits under the three scenarios. Of course, the FB profits are always the highest. MW is better than AQ only when the lead time is long (the even-numbered examples). This is intuitive because when the lead time increases, the benefit of demand smoothing becomes greater. This benefit is shown in Figure 2: MW is able to lower the holding and back-order costs from the status quo, but not as much as FB can. Thus the unobservability of selling effort hinders the firm's ability to induce the sales agent to smooth demand. Figure 3 shows the amount of selling effort exerted by the sales agent under the three scenarios. The agent works harder under FB than she does under the other two scenarios. It is interesting to note that whenever MW outperforms AQ (in the even-numbered examples), the sales agent expends more effort in MW than in AQ. So, does MW improve on AQ by extracting more surplus from the agent or by demand smoothing? Figure 4 answers this question, where *net benefit from sales force* (NBFS) is the firm's gross revenue from selling effort, i.e., $(p - c) * e$, where e is the expected annual effort minus the expected annual compensation received by the sales agent. Figure 4 shows the change in NBFS and the decrease in inventory costs, as the firm switches from AQ to MW. Interestingly, MW often leads to lower NBFS and always leads to lower inventory costs. Whenever the latter outweighs the former, MW is better than AQ. This suggests that MW improves upon AQ via demand smoothing, not by extracting more surplus from the sales force. Finally, Figure 5 shows that to reach FB, the firm must improve in both dimensions: NBFS as well as demand smoothing. Exactly how this can be done remains an open question.

Table 1 Numerical Examples.

No.	1	2	3	4	5	6	7	8
U_0	5	5	5	5	10	10	10	10
h	0.5	0.5	1	1	0.5	0.5	1	1
L	1	4	1	4	1	4	1	4

7. Conclusion

The thesis of this article is that in designing a compensation package for its sales force, the firm ought to consider the impact of the sales-force behavior on its product-delivery system. Generally speaking, a sales pattern that exhibits large swings causes difficulties in production and distribution planning. It is therefore beneficial to induce the sales force to exert selling effort in a way that smoothes the demand process. To achieve this, the compensation package should have the following features. 1) Incentive earnings (commissions and/or bonuses) should be based on the sales performance in a time window whose length is determined by the replenishment lead time. For example, if it takes two months to replenish the finished-goods inventory, then incentive earnings should be based on quarterly sales. 2) After taking into account the administrative costs and the psychological impact on the sales force, performance evaluation should be conducted as frequently as possible on a moving-time-window basis. For example, the sales force is evaluated every month based on its performance in the most recent quarter.[9] This strategy can be beneficial to firms with long lead times and significant costs associated with supply-demand mismatch.

The sales-force incentive problem naturally lies in the marketing-operations interface, and it should be studied as such. It is our hope that future research will continue to investigate the different facets of this problem.

Appendix: Omitted Proofs

Proof of Lemma 1. Take any values of $\hat{\xi}_K$ and ξ_K. For any $\delta > 0$, since $(\hat{\xi}_K + \xi_K + e + \delta - q)^+ \le (\hat{\xi}_K + \xi_K + e - q)^+ + \delta$ and $U(\cdot)$ is increasing, we have

[9] An executive from Hewlett-Packard Company once suggested that salespeople should receive incentive earnings on their birthdays based on the performance in the most recent year. This is an interesting way to achieve demand smoothing with a multiperson sales force. The author thanks Warren Hausman for this information.

Figure 1 **Firm Profits and Incentive Scenarios**

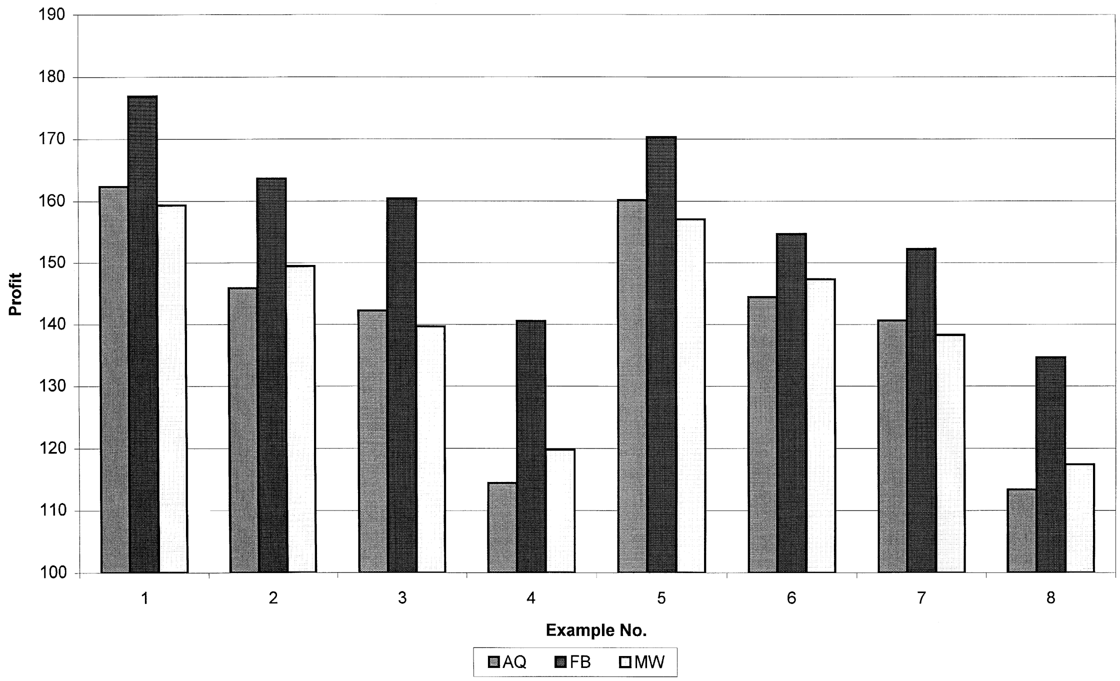


Figure 2 **Inventory Costs and Incentive Scenarios**

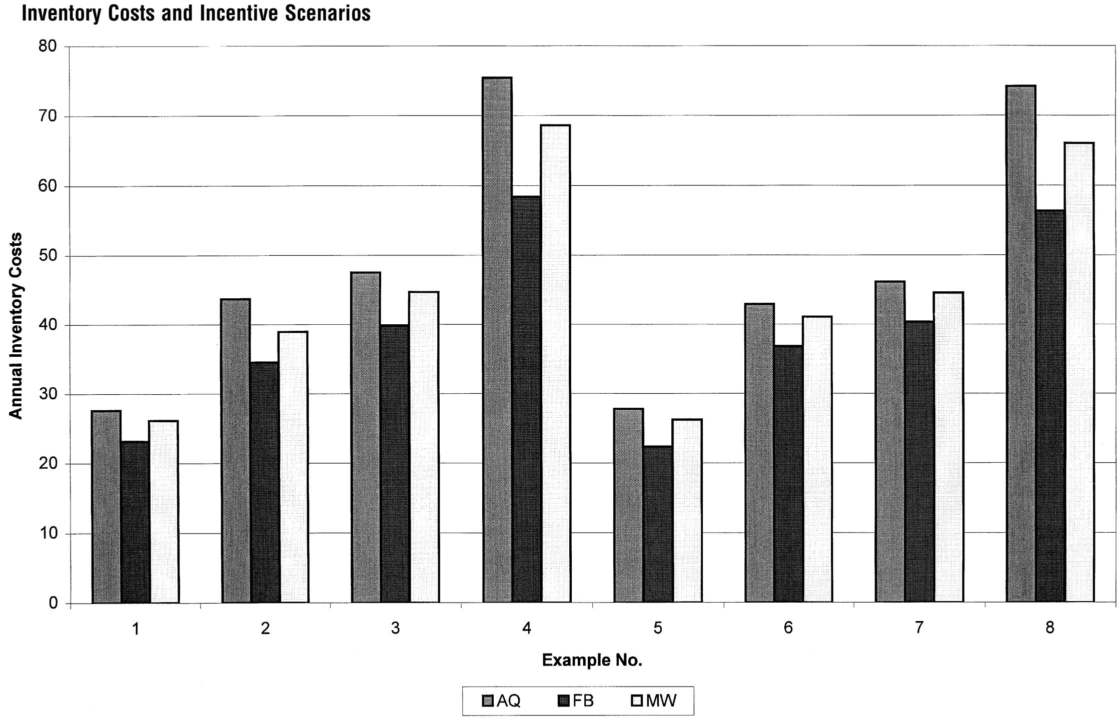


Figure 3 **Selling Effort and Incentive Scenarios**

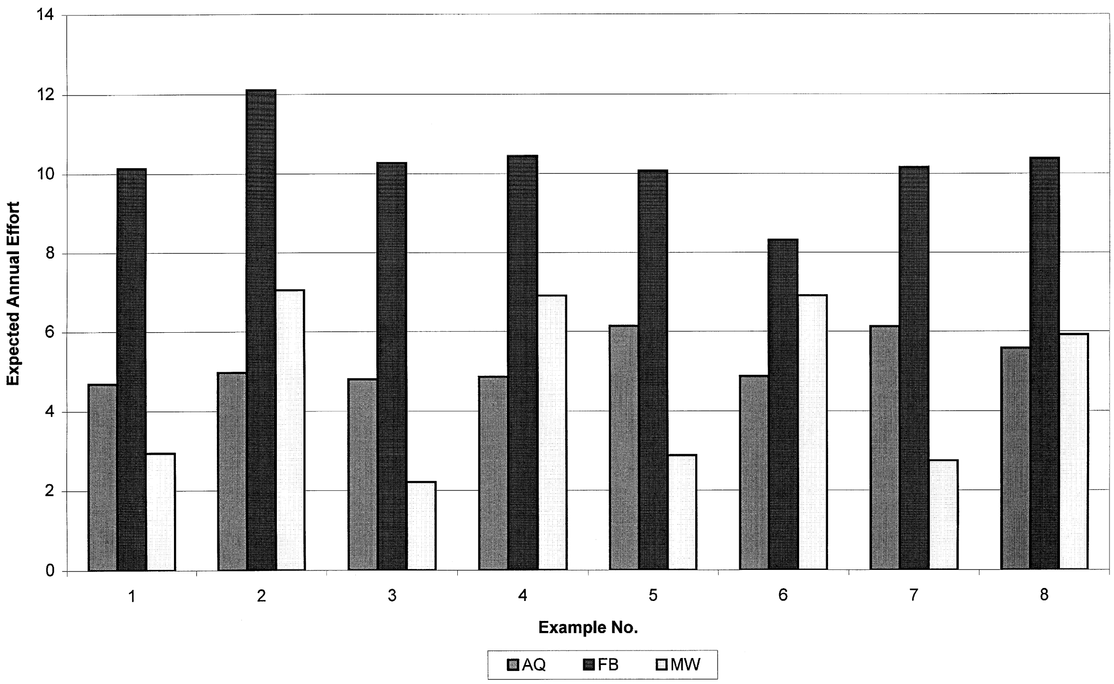


Figure 4 **From AQ to MW: Sources of Improvement**

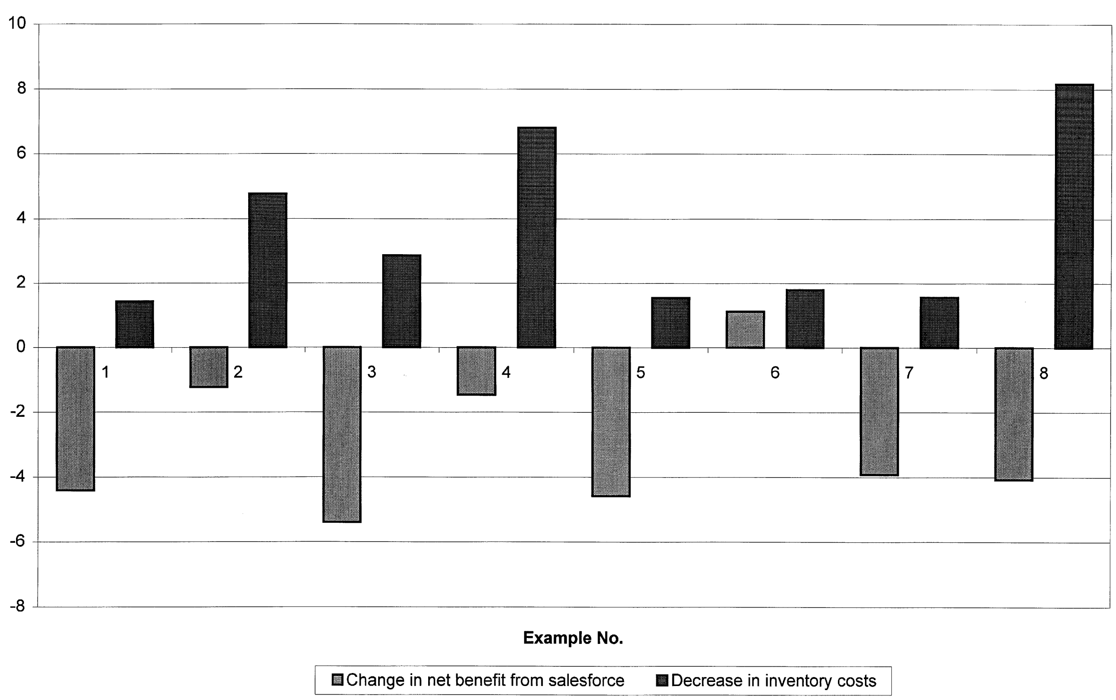
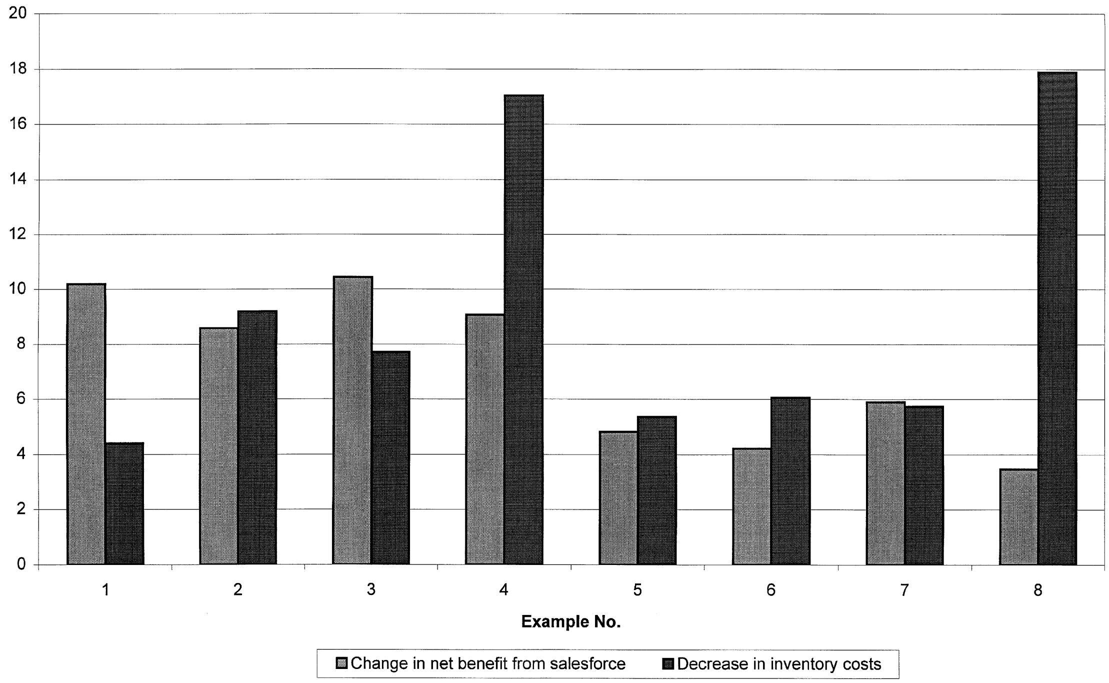


Figure 5 From AQ to FB: Sources of Improvement



$$U(\alpha + \beta(\hat{\xi}_K + \xi_K + e + \delta - q)^+)$$
$$\le U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+ + \beta\delta).$$

Therefore,

$$\begin{aligned} &U(\alpha + \beta(\hat{\xi}_K + \xi_K + e + \delta - q)^+) \\ &\quad - U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+) \\ &\le U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+ + \beta\delta) \\ &\quad - U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+) \\ &\le U(\alpha + \beta(\hat{\xi}_K + \xi_K + e + \delta - q)) \\ &\quad - U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)), \end{aligned}$$

where the second inequality follows since $U(\cdot)$ is concave and $(\hat{\xi}_K + \xi_K + e - q) \le (\hat{\xi}_K + \xi_K + e - q)^+$. Therefore, $\partial E\xi_K U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q)^+)/\partial e \le \partial E\xi_K U(\alpha + \beta(\hat{\xi}_K + \xi_K + e - q))/\partial e$. The lemma follows. □

Proof of Lemma 2. Define $a = q - \hat{\xi}_K$. Thus the agent's objective function can be written as

$$\int_{a-e}^{+\infty} U(\alpha + \beta(x + e - a))f(x)dx + \int_{-\infty}^{a-e} U(\alpha)f(x)dx - V(e).$$

Taking derivative of the objective function with respect to e and setting it to zero, we have the first-order condition:

$$\beta \int_{a-e}^{+\infty} U'(\alpha + \beta(x + e - a))f(x)dx - V'(e) = 0. \tag{14}$$

Suppose e is a local maximum. Then, the second-order derivative is less than or equal to zero, i.e.,

$$\beta^2 \int_{a-e}^{+\infty} U''(\alpha + \beta(x + e - a))f(x)dx + \beta U'(\alpha)f(a - e) - V''(e) \le 0. \tag{15}$$

Differentiating the left-hand side of Equation (14) with respect to α and rearranging terms, we have

$$\begin{aligned} &\frac{\partial e}{\partial \alpha}\Big[\beta^2 \int_{a-e}^{+\infty} U''(\alpha + \beta(x + e - a))f(x)dx \\ &\quad + \beta U'(\alpha)f(a - e) - V''(e)\Big] \\ &= -\beta \int_{a-e}^{+\infty} U''(\alpha + \beta(x + e - a))f(x)dx \\ &> 0, \end{aligned}$$

where the inequality follows since $U(\cdot)$ is strictly concave. The lemma follows by applying Equation (15). □

Proof of Theorem 2. Suppose $L = 0$. Let y_0 be a minimum point of $E[h(y - \xi_t)^+ + b(y - \xi_t)^-]$, a convex function of y. From Equation (1), $y^o(s) = y_0$ for all $s = (k, z) \in S$ with $k < K$, and $y^o(s) = y_0 + e(z)$ for all $s = (K, z) \in S$. It suffices to show that under the myopic base-stock policy, $IP(t) = y^o(s)$ for any t with $S_t = s$. Take any period t

with $S_t = (K, z)$ for some $z \geq 0$. Suppose $IP(t) = y_0 + e(z)$, the myopic base-stock level for period t. Since $D_t = \xi_t + e(z)$, the inventory position at the beginning of period $t + 1$ before ordering is $y_0 + e(z) - D_t \leq y_0$. Thus $IP(t + 1) = y_0$ under the myopic base-stock policy. It is then clear that $IP(t + 2) = \cdots = IP(t + K - 1) = y_0$, which implies that $IP(t + K) = y_0 + e(z')$ if $S_{t+K} = (K, z')$. □

Proof of Lemma 3. The lemma holds for $k = K$; see Equation (2). Now suppose it holds for $k + 1$. Take any z with $(k, z) \in S$. Since $(k + 1, z + \xi_k) \in S$ for any value of ξ_k, the inductive assumption implies that $EH_{k+1}(y - \xi_k, z + \xi_k)$ is convex in y. Recall that $G(y \mid (k, z))$ is convex in y. From Equation (3), the lemma holds for k. □

Proof of Theorem 3. Suppose $K = 2$. Thus $S = \{(1, 0), (2, z), z \geq 0\}$. Clearly, Equation (5) holds for state (1, 0). Now take any state $(2, z) \in S$. Note that $y^d(2, z) = y^o(2, z)$ and $(D[2, K] \mid (2, z)) \geq e(z)$. Thus Equation (5) holds for state $(2, z)$ if $y^o(2, z) - e(z) \leq y^d(1, 0)$. Given $S_t = (2, z)$, $D[t, t + L]$ is equal to $e(z)$ plus a random component, denoted by D^2. That is, $D[t, t + L] = e(z) + D^2$. Define $g(y) = E[h(y - D^2)^+ + b(y - D^2)^-]$, a convex function minimized at, say, y^2. Note that $G(y \mid (2, z)) = g(y - e(z))$ and thus $y^o(2, z) = y^2 + e(z)$. Now it only remains to show $y^2 \leq y^d(1, 0)$. Recall that

$$y^d(1, 0) = \operatorname{argmin}_y [G(y \mid (1, 0)) + EH_2(y - \xi_1, \xi_1)] \tag{16}$$

and that

$$\begin{aligned} H_2(w, z) &= G(\max\{w, y^o(2, z)\} \mid (2, z)) \\ &= g(\max\{w, y^o(2, z)\} - e(z)) \\ &= g(\max\{w - e(z), y^2\}). \end{aligned}$$

Thus $H_2(y - \xi_1, \xi_1) = g(\max\{y - \xi_1 - e(\xi_1), y^2\})$, which, as a function of y, is flat for $y \leq \xi_1 + e(\xi_1) + y^2$ and is thus flat for $y \leq y^2$. Consequently, $EH_2(y - \xi_1, \xi_1)$ is flat for $y \leq y^2$. From Equation (16), to have $y^2 \leq y^d(1, 0)$, it suffices to have

$$y^2 \leq y^o(1, 0), \tag{17}$$

which we now verify. Let D^1 be $D[t, t + L]$ given $S_t = (1, 0)$. Thus $G(y \mid (1, 0)) = E[h(y - D^1)^+ + b(y - D^1)^-]$. If D^1 is stochastically larger than D^2, then Equation (17) holds. The exact relationship between D^1 and D^2 depends on whether L is an even or an odd number. If, say, $L = 1$, then

$$D^1 = \xi_1 + \xi_2 + e(\xi_1) \quad \text{and} \quad D^2 = \xi_2 + \xi_3.$$

The former is stochastically larger. On the other hand, if, say, $L = 2$, then

$$D^1 = \xi_1 + \xi_2 + e(\xi_1) + \xi_3 \quad \text{and} \quad D^2 = \xi_2 + \xi_3 + \xi_4 + e(\xi_3),$$

which are stochastically the same. □

Proof of Theorem 5. Let $\psi(\cdot)$ and $\Psi(\cdot)$ be, respectively, the p.d.f. and c.d.f. of X. Let $\tilde{\Psi}(\cdot)$ be the c.d.f. of $X + \theta(X)$.

Theorem 5(i): Because $X + (B - X)^+ \geq B$, $\Pr(X + (B - X)^+ \leq x) = 0$ for any $x < B$. Thus Theorem 5 (i) is true for $z < B$. Now suppose $z \geq B$. Because for any $x \geq B$, $\Pr(X + (B - X)^+ \leq x) = \Psi(x)$, it suffices to show

$$\int_B^z \Psi(x)dx \leq \int_0^z \tilde{\Psi}(x)dx. \tag{18}$$

Because $X + \theta(X)$ is stochastically larger than X, $\tilde{\Psi}(x) \leq \Psi(x)$ for all x. Thus

$$\int_z^\infty [1 - \Psi(x)]dx \leq \int_z^\infty [1 - \tilde{\Psi}(x)]dx. \tag{19}$$

Because $E\theta(X) = E(B - X)^+ = \int_{x=0}^B \Psi(x)\,dx$, $E[X + \theta(X)] = E[X + (B - X)^+]$ can be written as

$$\int_0^\infty [1 - \tilde{\Psi}(x)]dx = \int_0^\infty [1 - \Psi(x)]dx + \int_0^B \Psi(x)dx.$$

The right side of the above equation can also be written as

$$\begin{aligned} &\int_z^\infty [1 - \Psi(x)]dx + \int_0^z [1 - \Psi(x)]dx + \int_0^B \Psi(x)dx \\ &= \int_z^\infty [1 - \Psi(x)]dx + z - \int_B^z \Psi(x)dx. \end{aligned}$$

Therefore,

$$\int_0^\infty [1 - \tilde{\Psi}(x)]dx = \int_z^\infty [1 - \Psi(x)]dx + z - \int_B^z \Psi(x)dx, \tag{20}$$

which implies

$$\begin{aligned} \int_0^z [1 - \tilde{\Psi}(x)]dx &= \int_0^\infty [1 - \tilde{\Psi}(x)]dx - \int_z^\infty [1 - \tilde{\Psi}(x)]dx \\ &\overset{(20)}{=} \int_z^\infty [1 - \Psi(x)]dx + z - \int_B^z \Psi(x)dx - \int_z^\infty [1 - \tilde{\Psi}(x)]dx \\ &\overset{(19)}{\leq} z - \int_B^z \Psi(x), \end{aligned}$$

which implies Equation (18).

Theorem 5(ii): Because $E[X + \theta(X)] = E[X + (B - X)^+]$, it suffices to show $E[X + \theta(X)]^2 \geq E[X + (B - X)^+]^2$. Note that

$$\begin{aligned} E[X + \theta(X)]^2 &= \int_0^\infty (x + \theta(x))^2 \psi(x)dx = \int_0^\infty x^2 \psi(x)dx \\ &\quad + \int_0^\infty 2x\theta(x)\psi(x)dx + \int_0^\infty \theta^2(x)\psi(x)dx \end{aligned}$$

and that

$$\begin{aligned} E[X + (B - X)^+]^2 &= \int_0^B B^2 \psi(x)dx + \int_B^\infty x^2 \psi(x)dx \\ &= B^2 \Psi(B) + \int_B^\infty x^2 \psi(x)dx. \end{aligned}$$

Therefore, it suffices to show

$$\int_0^B x^2 \psi(x)dx + \int_0^\infty 2x\theta(x)\psi(x)dx + \int_0^\infty \theta^2(x)\psi(x)dx \geq B^2 \Psi(B). \tag{21}$$

Because $E[X + (B - X)^+] = B\Psi(B) + \int_B^\infty x\psi(x)dx$ and $E[X + \theta(X)] = \int_0^\infty (x + \theta(x))\psi(x)dx$, we have from $E[X + \theta(X)] = E[X + (B - X)^+]$

$$B\Psi(B) = \int_0^B x\psi(x)dx + \int_0^\infty \theta(x)\psi(x)dx. \tag{22}$$

Note that

$$\int_B^\infty \theta(x)\psi(x)dx + 2\int_0^B (x + \theta(x))\psi(x)dx$$
$$= 2\int_0^B x\psi(x)dx + 2\int_0^\infty \theta(x)\psi(x)dx - \int_B^\infty \theta(x)\psi(x)dx$$
$$\overset{(22)}{=} 2B\Psi(B) - \int_B^\infty \theta(x)\psi(x)dx \le 2B\Psi(B) \le 2x\Psi(B)$$

for all $x \ge B$. Therefore,

$$\int_B^\infty \left\{2x + \theta(x) - \frac{\int_B^\infty \theta(x)\psi(x)dx + 2\int_0^B (x + \theta(x))\psi(x)dx}{\Psi(B)}\right\} \theta(x)\psi(x)dx \ge 0$$

because the integrand is nonnegative. After some algebra, the above inequality implies

$$\int_0^B (x + \theta(x))^2\psi(x)dx + \int_B^\infty 2x\theta(x)\psi(x)dx + \int_B^\infty \theta^2(x)\psi(x)dx$$
$$\ge \int_0^B \left(x + \theta(x) + \frac{1}{\Psi(B)}\int_B^\infty \theta(x)\psi(x)dx\right)^2 \psi(x)dx. \tag{23}$$

Note that the right side of Equation (23) is

$$\ge \Psi(B)\left(\int_0^B (x + \theta(x) + \frac{1}{\Psi(B)}\int_B^\infty \theta(x)\psi(x)dx)\frac{\psi(x)}{\Psi(B)}dx\right)^2$$
$$= \Psi(B)\left(\frac{1}{\Psi(B)}\int_0^B x\psi(x)dx + \frac{1}{\Psi(B)}\int_0^B \theta(x)\psi(x)dx + \frac{1}{\Psi(B)}\int_B^\infty \theta(x)\psi(x)dx\right)^2$$
$$= \frac{1}{\Psi(B)}\left(\int_0^B x\psi(x)dx + \int_0^\infty \theta(x)\psi(x)dx\right)^2$$
$$\overset{(22)}{=} \frac{1}{\Psi(B)}(B\Psi(B))^2 = B^2\Psi(B),$$

where the first inequality follows since w^2 is convex in w. The first part of Theorem 5(ii) follows by noting that the left side of Equation (23) is equal to the left side of Equation (21).

The second part of Theorem 5(ii) follows since

$$\mathrm{Var}[X + (B - X)^+] = E[X + (B - X)^+]^2 - (E[X + (B - X)^+])^2$$
$$= B^2\Psi(B) + \int_B^\infty x^2\psi(x)dx - (B\Psi(B) + \int_B^\infty x\psi(x)dx)^2$$

and

$$\frac{d\mathrm{Var}}{dB} = 2\Psi(B)\left[B - (B\Psi(B) + \int_B^\infty x\psi(x)dx)\right]$$
$$= 2\Psi(B)\,[B - E[X + (B - X)^+]]$$
$$\le 0. \quad \square$$

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