

STATIONARY POLICIES IN MULTIECHELON INVENTORY SYSTEMS WITH DETERMINISTIC DEMAND AND BACKLOGGING

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A stationary policy conducts replenishment activities—the placement and fulfillment of orders—in a stationary fashion. That is, each facility receives a constant batch (facility specific) in equal time intervals (facility specific) under a stationary policy. Although the advantages of stationary policies are clear (i.e., smooth operations), they represent a restriction in policy selection. This paper investigates how costly this restriction can be. For two multiechelon systems (serial and distribution) with deterministic demand and backlogging, we show that stationary policies are 70%-effective. This bound is tight in the sense that an example exists where the bound is reached. On the other hand, the average effectiveness of stationary policies is very high. In a set of 1,000 randomly generated numerical examples, we observed that the average effectiveness was 99%, and the standard deviation was 1.5%. The numerical examples also suggest that the performance of stationary policies deteriorates in systems where the setup cost decreases dramatically from an upstream stage to a downstream stage. Finally, a key building block of the above results is the existing lower bounds on the average costs of all feasible policies in the above systems. We provide a simpler derivation of these bounds.

Most companies schedule their operations in regular intervals so that the same set of activities is repeated, say, every week. In a production/distribution system, this means that the key replenishment activities—the placement and fulfillment of orders—are conducted in a stationary fashion. For example, a manufacturing plant releases a constant batch to the shop floor every month in order to create a stable production environment (see, e.g., Muckstadt and Roundy 1993). A stationary policy is one under which each facility receives a constant batch (facility specific) in equal time intervals (facility specific). Although stationary policies have clear managerial advantages, they represent a restriction in terms of policy selection. The main objective of this paper is to investigate how costly this restriction can be.

For a wide range of multiechelon inventory systems with deterministic demand, there exist easily computable policies that are guaranteed to be within 2% of optimal, or 98%-effective policies for short. (See, e.g., Roundy 1985 and 1986, Jackson et al. 1985, Maxwell and Muckstadt 1985, Mitchell 1987, Atkins et al. 1992, Federgruen et al. 1992, Federgruen and Zheng 1992, Atkins and Sun 1995, and Sun and Atkins 1997.) Some of the 98%-effective policies are stationary, and some are not. In this paper, we consider two multiechelon systems whose 98%-effective policies are not stationary.

Consider the serial system of Atkins and Sun (1995). In this system there are N stages arranged in series where customer demand occurs at stage 1; stage 1 orders from stage 2, 2 from 3, etc.; and stage N orders from an outside supplier with infinite stock. Each inventory transfer, either from the outside supplier or within the system, incurs a fixed setup cost. Customer demand arrives at a constant rate and is backlogged when stage 1 does not have any

on-hand inventory. Standard linear holding and backorder costs are assumed. Atkins and Sun identify a 98%-effective policy that requires one reorder interval at stage N , two different reorder intervals at stage $N - 1, \dots$, and N different reorder intervals at stage 1. Thus the policy is nonstationary. The high cost effectiveness is achieved by skillfully coordinating the reorder intervals at different stages. This need for careful coordination makes the policy difficult to implement.

On the other hand, the 98%-effective policy established in Mitchell (1987) is also nonstationary. Mitchell's system is one where a central warehouse replenishes a number of retailers, which in turn satisfy customer demand. The warehouse orders from an outside supplier with unlimited stock. The demand at each retailer arrives at a constant rate, and is backlogged when the retailer runs out of stock. The standard cost structure is assumed: linear holding costs at all facilities, linear backorder costs at the retailers, and fixed setup costs for inventory transfers from the outside supplier to the warehouse and from the warehouse to the retailers.

For the serial system, we establish that stationary policies are 70%-effective. This bound is tight in the sense that an example exists where the bound is reached. On the other hand, the average effectiveness of stationary policies is very high. In a set of 1,000 randomly generated numerical examples, we observed that the average effectiveness was 99%, and the standard deviation was only 1.5%. The numerical examples also suggest that the performance of stationary policies deteriorates in systems where the setup cost decreases dramatically from an upstream stage to a downstream stage.

We establish parallel results for the one-warehouse multi-retailer system. We show that integer-ratio policies are

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70%-effective. An integer-ratio policy is one where (1) the warehouse orders in equal intervals; (2) each retailer receives a constant batch in equal intervals; and (3) the replenishment interval at the warehouse, denoted by T , and the replenishment interval at retailer n , denoted by T_n , satisfy the integer-ratio constraint and either T/T_n or T_n/T is a positive integer. Notice that the warehouse is not required to order the same quantity every time it places an order. Therefore, an integer-ratio policy may not be stationary at the warehouse level, but it is stationary at the retail level, which is the key difference between integer-ratio policies and the 98%-effective policy by Mitchell. (Mitchell called his policy nearly integer-ratio.) The reason for relaxing the stationarity requirement at the warehouse is that if we also required constant order quantities at the warehouse, then the policy would become a nested policy and as Roundy (1985) points out, nested policies could be arbitrarily bad.

A key building block of the above results is the existing lower bounds on the average costs of all feasible policies in the serial and the one-warehouse multiretailer systems. These bounds are established in Atkins and Sun (1995) and Mitchell (1987). A separate contribution of this paper is a simpler derivation of these bounds. The new proof is based on cost allocation.

The rest of the paper is organized as follows. Section 1 considers the serial system. Section 2 deals with the one-warehouse multiretailer system. Concluding remarks are in Section 3. The new derivation of the lower bounds is described in Appendix A.

1. SERIAL SYSTEMS

1.1. Model and Notation

Consider a single-item, serial system with N stages where stage 1 orders from stage 2, 2 from 3, etc., and stage N orders from an outside supplier with unlimited stock. Each shipment to a stage incurs a fixed setup cost. We assume that the transportation leadtimes at all stages are zero. (The case with positive constant leadtimes is essentially the same.) Customer demand arrives only at stage 1 at a constant rate. For easy presentation, we assume that the demand rate is two units per unit of time. When stage 1 runs out of stock, demand is backlogged. We assume linear holding and backorder costs. The objective is to minimize the average total cost over the infinite horizon. Define:

$$\begin{aligned} K_i &= \text{setup cost at stage } i, \\ h_i &= \text{echelon holding cost rate at stage } i, \\ p &= \text{backorder cost rate (at stage 1)}, \\ H_i &= \text{installation holding cost rate at stage } i = \sum_{j=i}^N h_j, \\ H_0 &= pH_1/(p + H_1), \\ \alpha_i &= (p + H_{i+1})/(p + H_i) \text{ with } H_{N+1} = 0, \text{ and} \\ \beta_i &= p^2/[(p + H_i)(p + H_{i+1})]. \end{aligned}$$

1.2. Stationary Policies

Under a stationary policy, stage i receives a constant batch from stage $i + 1$ every T_i units of time, $i = 1, \dots, N$.

(Stage $N + 1$ is the outside supplier.) Since the demand rate is 2, the size of the constant batch received by stage i is $2T_i$. Note that the optimal stationary policy must be nested: whenever a stage receives a shipment, it sends a batch to its downstream stage. (For example, see Schwarz 1973.) Thus, $T_i = m_i T_{i-1}$ for $i = 2, \dots, N$, where m_i is a positive integer. Also note that the optimal stationary policy must have the zero-inventory-ordering property at stage $i \geq 2$: stage i has zero on-hand inventory just prior to each delivery to the stage (see, e.g., Atkins and Sun 1995). Therefore, we concentrate on stationary policies with the above two properties. We will refer to such policies as stationary policies.

1.3. Average Costs

First consider the installation holding costs at stage $i \geq 2$. At the beginning of its replenishment cycle, stage i receives a batch of size $2T_i$. Just prior to receiving the batch, stage i has zero on-hand inventory. Since the batch is sent to stage $i - 1$ in m_i equal installments, the on-hand inventory at stage i follows a staircase path. Therefore, the total installation holding cost at stage i in the cycle is $H_i(T_i^2 - m_i T_{i-1}^2)$, and the average is

$$H_i(T_i^2 - m_i T_{i-1}^2)/T_i = H_i(T_i - T_{i-1}), \quad i = 2, \dots, N.$$

Now consider the installation holding costs and backorder costs at stage 1. Note that stage 1 is essentially a single-stage system with holding cost rate H_1 and backorder cost rate p . From the EOQ model with backlogging, we know that since the cycle length at stage 1 is T_1 , the minimum average holding and backorder costs at stage 1 are $H_0 T_1$, assuming that stage 1 uses the optimal reorder point. Finally, the average setup cost at stage i is K_i/T_i . Thus the average total cost in the system is

$$C^s(\bar{T}) \stackrel{\text{def}}{=} \sum_{i=2}^N [K_i/T_i + h_i T_i] + [K_1/T_1 + (H_0 - H_2)T_1],$$

where $\bar{T} \stackrel{\text{def}}{=} (T_1, \dots, T_N)$. An optimal stationary policy minimizes the above cost function subject to the integer-ratio constraint: $T_i = m_i T_{i-1}$, $i = 2, \dots, N$, where m_i is a positive integer.

Notice that the coefficient of T_1 in the above cost function may not be positive. Suppose that this is the case. It is easy to see that the optimal stationary policy must have $T_1 = T_2$. Thus T_1 in the cost function can be replaced with T_2 , reducing one variable. The coefficient of T_2 is now $(H_0 - H_3)$. If $H_0 - H_3 \leq 0$, then replace T_2 with T_3 , reducing one more variable. Continue in this fashion until all the coefficients in the cost function are positive. The resulting cost function has exactly the same form as the cost function of a serial system without backorders. For the remainder of this subsection, we assume that all the coefficients in the original cost function are positive. (If this is untrue, just replace $C^s(\bar{T})$ with the new cost function with all positive coefficients, and everything holds.)

Define

$$B^s = \min C^s(\bar{T}),$$

$$\text{s.t. } T_{i-1} \leq T_i, \quad i = 2, \dots, N$$

$$T_i > 0, \quad i = 1, \dots, N.$$

Thus B^s is a lower bound on the average costs of all feasible stationary policies. Since $C^s(\bar{T})$ has the form of the cost function of a serial system without backorders, the above minimization problem can be solved easily by using an algorithm in Muckstadt and Roundy (1993). The solution can then be used to construct a power-of-two policy (a feasible stationary policy) with average cost C^s so that

$$B^s/C^s \geq \sqrt{2} \ln 2 \approx 98\%. \quad (1)$$

(For example, see Roundy 1985.)

1.4. Worst-Case Analysis

Define $C_i(T) = K_i/T + \beta_i h_i T$, $i = 1, \dots, N$. Define

$$B^* = \min \sum_{i=1}^N C_i(T_i)$$

$$\text{s.t. } T_{i-1} \leq T_i, \quad i = 2, \dots, N$$

$$T_i > 0, \quad i = 1, \dots, N.$$

As shown in Atkins and Sun (1995), B^* is a lower bound on the average costs of all feasible policies in the serial system. (A new proof of this bound is described in Appendix A.) It turns out that $B^*/B^s \geq 1/\sqrt{2}$, which together with (1) leads to $B^*/C^s \geq \ln 2 \approx 70\%$. In other words, stationary policies are 70%-effective. Since the N -stage case is rather cumbersome to present, we concentrate on the two-stage case below. (The N -stage case is proved in Chen 1994a).

Consider the serial system with $N = 2$. In this case

$$B^* = \min_{T_1 \leq T_2} [K_2/T_2 + \beta_2 h_2 T_2] + [K_1/T_1 + \beta_1 h_1 T_1].$$

Using an algorithm in Muckstadt and Roundy (1993), we have

Scenario (i):

$$K_2/(\beta_2 h_2) > K_1/(\beta_1 h_1):$$

$$B^* = 2\sqrt{K_2 \beta_2 h_2} + 2\sqrt{K_1 \beta_1 h_1};$$

Scenario (ii):

$$K_2/(\beta_2 h_2) \leq K_1/(\beta_1 h_1):$$

$$B^* = 2\sqrt{(K_1 + K_2)(\beta_2 h_2 + \beta_1 h_1)}$$

$$= 2\sqrt{(K_1 + K_2)H_0},$$

since $\beta_2 h_2 + \beta_1 h_1 = H_0$. Similarly,

$$B^s = \min_{T_1 \leq T_2} [K_2/T_2 + h_2 T_2] + [K_1/T_1 + (H_0 - h_2)T_1].$$

Thus

Scenario (iii):

$$H_0 > h_2 \text{ and } K_2/h_2 \geq K_1/(H_0 - h_2):$$

$$B^s = 2\sqrt{K_2 h_2} + 2\sqrt{K_1(H_0 - h_2)};$$

Scenario (iv):

$$H_0 > h_2 \text{ and } K_2/h_2 < K_1/(H_0 - h_2):$$

$$B^s = 2\sqrt{(K_1 + K_2)H_0};$$

Scenario (v):

$$H_0 \leq h_2: B^s = 2\sqrt{(K_1 + K_2)H_0}.$$

(Under scenarios (iii) and (iv), the minimization problem for B^s has the same structure as the problem for B^* . In scenario v, the coefficient of T_1 , $H_0 - h_2$ is not positive and thus the optimal solution must have $T_2 = T_1$. In this case, the problem has effectively only one variable.)

Lemma 1. $\max\{\theta, \beta_2\} \geq 1/2$, where $\theta \stackrel{\text{def}}{=} \beta_2 h_2/H_0$.

Proof. If $p \geq h_2$, then the lemma holds, since $\beta_2 = p/(p + h_2) \geq 1/2$. On the other hand, if $p < h_2$, then since $p > H_0$ by definition, we have $\theta > h_2/(p + h_2) > 1/2$ and so the lemma. \square

Theorem 1. $B^*/B^s \geq 1/\sqrt{2}$.

Proof. First note that for any $a, b, c, d > 0$, we have $a/b \geq (>) c/d$ if and only if $a/b \geq (>) (a + c)/(b + d) \geq (>) c/d$. Consider the following three cases.

Case 1. $\beta_2 h_2 \geq K_2 H_0/(K_1 + K_2)$. Since $\beta_2 h_2 \geq K_2 H_0/(K_1 + K_2)$ and $H_0 = \beta_2 h_2 + \beta_1 h_1$, $K_2/(\beta_2 h_2) \leq K_1/(\beta_1 h_1)$. Thus Scenario (ii) holds, and $B^* = 2\sqrt{(K_1 + K_2)H_0}$. On the other hand, if Scenario (iii) holds, then $K_2/h_2 \geq (K_1 + K_2)/H_0$ or $h_2 \leq K_2 H_0/(K_1 + K_2)$, which is impossible under this case since $\beta_2 < 1$. Consequently, only Scenarios (iv) or (v) are possible, and $B^s = 2\sqrt{(K_1 + K_2)H_0}$. Therefore, $B^*/B^s = 1 \geq 1/\sqrt{2}$.

Case 2. $h_2 \leq K_2 H_0/(K_1 + K_2)$. Since $h_2 \leq K_2 H_0/(K_1 + K_2)$ and $\beta_2 < 1$, we have $\beta_2 h_2 < K_2 H_0/(K_1 + K_2)$ or $K_2/(\beta_2 h_2) > (K_1 + K_2)/H_0$. Since $H_0 = \beta_2 h_2 + \beta_1 h_1$, $K_2/(\beta_2 h_2) > K_1/(\beta_1 h_1)$. Thus Scenario (i) occurs, and $B^* = 2\sqrt{K_2 \beta_2 h_2} + 2\sqrt{K_1 \beta_1 h_1}$. On the other hand, since $h_2 \leq K_2 H_0/(K_1 + K_2)$, we have $h_2 < H_0$ and $K_2/h_2 \geq (K_1 + K_2)/H_0$ or $K_2/h_2 \geq K_1/(H_0 - h_2)$. Thus Scenario (iii) occurs, and $B^s = 2\sqrt{K_2 h_2} + 2\sqrt{K_1(H_0 - h_2)}$. Therefore,

$$\frac{B^*}{B^s} = \frac{2\sqrt{K_2 \beta_2 h_2} + 2\sqrt{K_1 \beta_1 h_1}}{2\sqrt{K_2 h_2} + 2\sqrt{K_1(H_0 - h_2)}}$$

$$= \frac{\sqrt{K_2 \beta_2 h_2} + \sqrt{K_1(H_0 - \beta_2 h_2)}}{\sqrt{K_2 h_2} + \sqrt{K_1(H_0 - h_2)}} \geq \sqrt{\beta_2}.$$

Since $h_2 \leq K_2 H_0/(K_1 + K_2) < H_0 < p$, we have $\beta_2 > 1/2$. The theorem follows.

Case 3. $\beta_2 h_2 < K_2 H_0/(K_1 + K_2) < h_2$. In this case, Scenario (i) occurs, and either Scenario (iv) or (v) occurs. Therefore,

$$\frac{B^*}{B^s} = \frac{2\sqrt{K_2 \beta_2 h_2} + 2\sqrt{K_1 \beta_1 h_1}}{2\sqrt{(K_1 + K_2)H_0}}$$

$$= \frac{\sqrt{K_2 \beta_2 h_2} + \sqrt{K_1(H_0 - \beta_2 h_2)}}{\sqrt{(K_1 + K_2)H_0}}. \quad (2)$$

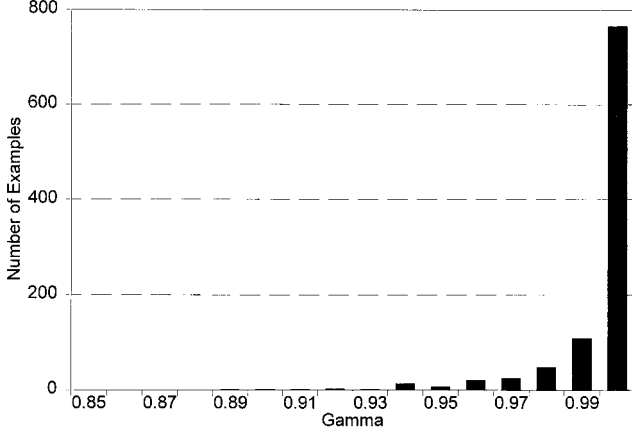


Figure 1. Frequency diagram of gamma.

From $\beta_2 h_2 < K_2 H_0 / (K_1 + K_2)$ and the definition of θ , we have $\theta < K_2 / (K_1 + K_2)$ or $(1 - \theta) > K_1 \theta / K_2$. Thus, from (2),

$$\frac{B^*}{B^s} = \frac{\sqrt{K_2 \theta} + \sqrt{K_1 (1 - \theta)}}{\sqrt{K_1 + K_2}} > \frac{\sqrt{K_2 \theta} + \sqrt{K_1^2 \theta / K_2}}{\sqrt{K_1 + K_2}} > \sqrt{\theta}. \quad (3)$$

On the other hand, from $\beta_2 h_2 < K_2 H_0 / (K_1 + K_2) < h_2$, we have

$$\begin{aligned} \sqrt{K_2 \beta_2 h_2} &> \sqrt{K_2 \beta_2 K_2 H_0 / (K_1 + K_2)} \\ &= K_2 \sqrt{\beta_2 H_0 / (K_1 + K_2)}, \end{aligned}$$

and

$$\begin{aligned} \sqrt{K_1 (H_0 - \beta_2 h_2)} &> \sqrt{K_1 (H_0 - K_2 H_0 / (K_1 + K_2))} \\ &= K_1 \sqrt{H_0 / (K_1 + K_2)}. \end{aligned}$$

Using the above two inequalities in (2), we have

$$\frac{B^*}{B^s} > \frac{K_1}{K_1 + K_2} + \frac{K_2}{K_1 + K_2} \sqrt{\beta_2} > \sqrt{\beta_2}, \quad (4)$$

since $\beta_2 < 1$. By (3) and (4), $B^*/B^s > \max\{\sqrt{\beta_2}, \sqrt{\theta}\}$. The theorem follows from Lemma 1. \square

Theorem 2. $B^*/C^s \geq \ln 2 \approx 70\%$.

Proof. Follows from Theorem 1 and (1). \square

1.5. Numerical Examples

We first use a set of randomly generated numerical examples to illustrate the average effectiveness of stationary policies. We then provide a sequence of examples that reach the worst-case bound in limit, suggesting that the bound is tight. Since $B^*/C^s \geq 98\%$, we focus on $\gamma \stackrel{\text{def}}{=} B^*/B^s$. (From Theorem 1, the worst-case bound on γ is $1/\sqrt{2}$.)

Consider the serial system with two stages. It has five cost parameters: K_1 , K_2 , h_1 , h_2 , and p . We randomly generated these parameters from the following uniform distributions: $K_1 \in U(0, 50)$, $K_2 \in U(0, 100)$, $h_1 \in U(0, 5)$, $h_2 \in U(0, 5)$, and $p \in U(0, 25)$. There were 1,000 examples. For each example we computed the γ value. Figure 1 shows the frequency diagram. The average γ value is 99%,

Table I
The Worst Five Examples

K_1	K_2	h_1	h_2	p	γ
0.93	99.94	2.98	3.46	6.48	0.88
2.33	82.17	3.07	1.43	3.73	0.89
0.62	70.88	4.24	2.59	8.30	0.89
0.08	62.39	1.66	3.87	9.93	0.90
1.29	83.04	2.49	2.78	4.35	0.90

and the standard deviation is 1.5%. This suggests that the average effectiveness of stationary policies is very high.

Table I lists five examples with the lowest γ values. All these examples have a small K_1/K_2 ratio, suggesting that stationary policies are costly for systems where the setup cost decreases dramatically from an upstream stage to a downstream stage.

We next provide a sequence of examples that reach the worst-case bound in limit. Consider a two-stage system with $K_1 = 1/n$, $K_2 = 1$, $h_1 = n + 1/n$, $h_2 = 1 - 1/n$, and $p = 1 + 1/n$, where $n \geq 2$ is an integer. Check that $H_1 = n + 1$, $H_0 = p H_1 / (p + H_1) = 1$, $\beta_2 h_2 = (1 - 1/n^2)/2$, and $\beta_1 h_1 = (1 + 1/n^2)/2$. Since

$$\begin{aligned} \frac{K_2}{h_2} &= \frac{1}{1 - 1/n} > 1 = \frac{K_1}{H_0 - h_2} \quad \text{and} \\ \frac{K_2}{\beta_2 h_2} &= \frac{2}{1 - 1/n^2} > \frac{2/n}{1 + 1/n^2} = \frac{K_1}{\beta_1 h_1}, \end{aligned}$$

we have

$$B^s = 2 \sqrt{K_2 h_2} + 2 \sqrt{K_1 (H_0 - h_2)} = 2 \sqrt{1 - \frac{1}{n}} + \frac{2}{n},$$

and

$$\begin{aligned} B^* &= 2 \sqrt{K_2 \beta_2 h_2} + 2 \sqrt{K_1 \beta_1 h_1} \\ &= \frac{2}{\sqrt{2}} \sqrt{1 - \frac{1}{n^2}} + \frac{2}{\sqrt{2}} \sqrt{\frac{1}{n} \left(1 + \frac{1}{n^2}\right)}. \end{aligned}$$

As $n \rightarrow \infty$, $B^s \rightarrow 2$, $B^* \rightarrow 2/\sqrt{2}$; thus $B^*/B^s \rightarrow 1/\sqrt{2}$. (Note that as $n \rightarrow \infty$, $K_1/K_2 \rightarrow 0$.)

2. ONE-WAREHOUSE MULTIRETAILER SYSTEMS

2.1. Model and Notation

Consider a distribution system in which a central warehouse orders from an outside supplier with unlimited stock and replenishes N retailers. Each shipment from the outside supplier to the warehouse or from the warehouse to the retailers incurs a fixed setup cost. The transportation leadtimes are all assumed to be zero. (The case with positive constant leadtimes is basically the same.) Customer demand arrives at each retailer at a constant rate. When a retailer runs out of stock, demand is backlogged. We assume linear holding and backorder costs. The objective is to minimize the average total cost over the infinite horizon. For easy presentation, we assume that the demand rate at each retailer is two units per unit of time, and that

each retailer carries a different product, i.e., retailer n carries product n for $n = 1, \dots, N$. Define:

$$\begin{aligned} K_0 &= \text{fixed setup cost for each shipment to the warehouse,} \\ K_n &= \text{fixed setup cost for each shipment (of product } n \text{) to retailer } n, \\ h_{1n} &= \text{echelon holding cost rate for product } n \text{ at the retail level,} \\ h_{2n} &= \text{echelon holding cost rate for product } n \text{ at the warehouse level,} \\ p_n &= \text{backorder cost rate for product } n \text{ (at the retail level),} \\ H_n &= \text{installation holding cost rate for product } n \text{ at the retail level,} \\ &= h_{1n} + h_{2n} \\ H_{0n} &= p_n H_n / (p_n + H_n), \\ \alpha_n &= p_n / (p_n + h_{2n}), \text{ and} \\ \beta_n &= \alpha_n p_n / (p_n + H_n). \end{aligned}$$

2.2. Lower Bound

Let \vee be the maximum operator, i.e., $x \vee y = \max \{x, y\}$. Define

$$\begin{aligned} c_n(T, T_n) &= K_n/T_n + \alpha_n h_{2n}(T \vee T_n) + \beta_n h_{1n} T_n, \\ T > 0, T_n > 0, n &= 1, \dots, N. \end{aligned}$$

Let $b_n(T) = \min_{T_n > 0} c_n(T, T_n)$ and $B(T) = K_0/T + \sum_{n=1}^N b_n(T)$. Let $B(T)$ be minimized at T^* . Set $B^* = B(T^*)$, which is a lower bound on the average costs of all feasible policies in the one-warehouse N -retailer system (Mitchell 1987).

Following Roundy (1985), we express B^* in a convenient form. Define:

$$\tau_{1n} = \sqrt{\frac{K_n}{H_{0n}}}$$

and

$$\tau_{2n} = \sqrt{\frac{K_n}{\beta_n h_{1n}}}, \quad n = 1, \dots, N.$$

Thus $\tau_{1n} < \tau_{2n}$ since $H_{0n} = \alpha_n h_{2n} + \beta_n h_{1n} > \beta_n h_{1n}$. Define $G = \{n : T^* < \tau_{1n}\}$, $E = \{n : \tau_{1n} \leq T^* \leq \tau_{2n}\}$, and $L = \{n : \tau_{2n} < T^*\}$. It is easy to verify that $c_n(T^*, T_n)$ is minimized at $T_n = \tau_{1n}$ (resp., T^* , τ_{2n}) if $n \in G$ (resp., E , L). Therefore,

$$\begin{aligned} B^* &= \frac{K_0 + \sum_{n \in E} K_n}{T^*} + \left(\sum_{n \in L} \alpha_n h_{2n} + \sum_{n \in E} H_{0n} \right) T^* \\ &+ \sum_{n \in G} \left(\frac{K_n}{\tau_{1n}} + H_{0n} \tau_{1n} \right) + \sum_{n \in L} \left(\frac{K_n}{\tau_{2n}} + \beta_n h_{1n} \tau_{2n} \right). \quad (5) \end{aligned}$$

2.3. Integer-Ratio Policies

Under an integer-ratio policy, retailer n receives a constant batch from the warehouse every $T_n > 0$ units of time, beginning at time $t = 0$, $n = 1, \dots, N$, and the warehouse orders from the outside supplier every T units of time, beginning at time $t = 0$. Moreover, this policy (1) satisfies

the integer-ratio constraint, i.e., $T/T_n \in R$, where R is the set of all positive integers and their reciprocals, $n = 1, \dots, N$; and (2) has the zero-inventory-ordering property in the sense that the warehouse has zero on-hand inventory just prior to every delivery from the outside supplier. This is precisely the integer-ratio policy defined in Roundy (1985) for one-warehouse multiretailer systems without backorders.

Remark 1. If $T/T_n \geq 1$ for $n = 1, \dots, N$, then the integer-ratio policy is stationary. In this case, the warehouse receives a constant batch every T units of time. Now if $T_n/T > 1$ for some n , then the integer-ratio policy is not stationary. This is because the quantities delivered to the warehouse are no longer the same. As Roundy pointed out, integer-ratio policies that are stationary (also called nested policies) may be arbitrarily bad. As a result, we do not require stationarity at the warehouse level. Compared with the 98%-effective policy by Mitchell (1987), the advantage of integer-ratio policies is that the inventory flow from the warehouse to the retailers is stationary.

2.4. Average Costs

Consider product $n = 1, \dots, N$. For fixed T and T_n , let $F_n^s(T, T_n)$ be the minimum average holding and backorder costs associated with the product. Note that as far as product n is concerned, the distribution system is a two-stage serial system, with the warehouse being the upstream stage and the retailer the downstream stage. Therefore, if $T_n \leq T$, we have from subsection 1.3:

$$I_n^s(T, T_n) = h_{2n}T + (H_{0n} - h_{2n})T_n.$$

On the other hand, if $T_n > T$, then the warehouse does not hold any inventory (of product n), and the two-stage serial system is essentially a single-stage system. From the EOQ model with backlogging,

$$I_n^s(T, T_n) = H_{0n}T_n.$$

Combining the above two cases, we have

$$I_n^s(T, T_n) = h_{2n}(T \vee T_n) + (H_{0n} - h_{2n})T_n.$$

Define

$$c_n^s(T, T_n) = K_n/T_n + h_{2n}(T \vee T_n) + (H_{0n} - h_{2n})T_n.$$

Therefore, the average total cost in the system is

$$C^s(\bar{T}) = K_0/T + \sum_{n=1}^N c_n^s(T, T_n),$$

where $\bar{T} \stackrel{\text{def}}{=} (T, T_1, \dots, T_N)$. An optimal integer-ratio policy minimizes the average total cost subject to the integer-ratio constraint: $T/T_n \in R$, $n = 1, \dots, N$.

Define $b_n^s(T) = \min_{T_n > 0} c_n^s(T, T_n)$ and $B^s(T) = K_0/T + \sum_{n=1}^N b_n^s(T)$. Let $B^s(T)$ be minimized at $T = T^s$. Set $B^s = B^s(T^s)$. Thus B^s is a lower bound on the average costs of all feasible integer-ratio policies.

Since $C^s(\bar{T})$ has the form of the average total cost of integer-ratio policies in one-warehouse multiretailer systems without backorders (see Roundy 1985), one can use Roundy's approach to compute a power-of-two policy (a feasible integer-ratio policy) that is within 2% of B^s . Let C^s be the average total cost of this power-of-two policy. Thus,

$$B^s/C^s \geq \sqrt{2} \ln 2 \approx 98\%. \quad (6)$$

As in subsection 2.2, we express B^s in a convenient form. For $n = 1, \dots, N$, define

$$\tau_{1n}^s = \sqrt{\frac{K_n}{H_{0n}}} \quad \text{and}$$

$$\tau_{2n}^s = \begin{cases} \sqrt{K_n/(H_{0n} - h_{2n})}, & H_{0n} - h_{2n} > 0, \\ \infty, & H_{0n} - h_{2n} \leq 0. \end{cases}$$

Define $G^s = \{n : \tau_{1n}^s > T^s\}$, $E^s = \{n : \tau_{1n}^s \leq T^s \leq \tau_{2n}^s\}$, and $L^s = \{n : \tau_{2n}^s < T^s\}$. Thus if $n \in G^s$ (resp., E^s , L^s), then $c_n^s(T^s, T_n)$ is minimized at $T_n = \tau_{1n}^s$ (resp., T^s , τ_{2n}^s). Consequently,

$$B^s = \frac{K_0 + \sum_{n \in E^s} K_n}{T^s} + \left(\sum_{n \in L^s} h_{2n} + \sum_{n \in E^s} H_{0n} \right) T^s$$

$$+ \sum_{n \in G^s} \left(\frac{K_n}{\tau_{1n}^s} + H_{0n} \tau_{1n}^s \right)$$

$$+ \sum_{n \in L^s} \left(\frac{K_n}{\tau_{2n}^s} + (H_{0n} - h_{2n}) \tau_{2n}^s \right). \quad (7)$$

Moreover, from Roundy (1985),

$$\frac{K_0 + \sum_{n \in E^s} K_n}{(T^s)^2} = \sum_{n \in L^s} h_{2n} + \sum_{n \in E^s} H_{0n}. \quad (8)$$

2.5. Worst-Case Analysis

We first derive a lower bound on B^*/B^s . The idea is to decompose the one-warehouse N -retailer system to N two-stage serial systems by allocating the setup cost at the warehouse among the different products and by reallocating the echelon holding costs. This decomposition enables us to use the results described in Section 1. A similar technique is used in Atkins and Iyogun (1987).

For $n = 1, \dots, N$, define

$$h_{2n}^s = \begin{cases} h_{2n} & n \in L^s, \\ H_{0n} - K_n/(T^s)^2 & n \in E^s, \\ 0 & n \in G^s, \end{cases}$$

and

$$K_{0n} = \begin{cases} h_{2n}^s (T^s)^2 & n \in L^s \cup E^s, \\ 0 & n \in G^s. \end{cases}$$

Lemma 2. $h_{2n}^s \leq h_{2n}$, $n = 1, \dots, N$.

Proof. Take any $n = 1, \dots, N$. If $n \in L^s \cup E^s$, then the lemma follows by definition. Now suppose $n \in G^s$. If $h_{2n} < H_{0n}$ then from the definition of E^s ,

$$T^s \leq \tau_{2n}^s = \sqrt{\frac{K_n}{H_{0n} - h_{2n}}} \quad \text{or} \quad h_{2n} \geq H_{0n} - \frac{K_n}{(T^s)^2} = h_{2n}^s.$$

On the other hand, if $h_{2n} \geq H_{0n}$, then the lemma holds because $h_{2n}^s < H_{0n}$ by definition. \square

Lemma 3. $\sum_{n=1}^N K_{0n} = K_0$.

Proof. Follows from the definition of K_{0n} and (8). \square

As previously mentioned, as far as product n is concerned, the distribution system is a two-stage serial system, with the warehouse being the upper stage and retailer n the lower stage. Now allocate the following costs to this serial system:

$$K_{0n} = \text{setup cost for each shipment of the product to the warehouse,}$$

$$K_n = \text{setup cost for each shipment to retailer } n,$$

$$h_{2n}^s = \text{echelon holding cost rate at the warehouse,}$$

$$h_{1n}^s = \text{echelon holding cost rate at retailer } n = H_n - h_{2n}^s (>0 \text{ from Lemma 2}), \text{ and}$$

$$p_n = \text{backorder cost rate at retailer } n.$$

By Lemma 2, the above cost allocation does not change the installation holding cost rate at retailer n (i.e., it is still H_n), but it lowers the installation holding cost rate for product n at the warehouse from h_{2n} to h_{2n}^s . Together with Lemma 3, this establishes that the above cost allocation does not overcharge costs.

Consider the two-stage serial system of product n . Define $\alpha_n^s = p_n/(p_n + h_{2n}^s)$ and $\beta_n^s = \alpha_n^s p_n/(p_n + H_n)$. From Section 1:

$$B_n^* \stackrel{\text{def}}{=} \min_{T_2 \geq T_1 > 0} [K_{0n}/T_2 + \alpha_n^s h_{2n}^s T_2$$

$$+ K_n/T_1 + \beta_n^s h_{1n}^s T_1],$$

is a lower bound on the average costs of all feasible policies in the system. On the other hand,

$$B_n^s \stackrel{\text{def}}{=} \min_{T_2 \geq T_1 > 0} [K_{0n}/T_2 + h_{2n}^s T_2$$

$$+ K_n/T_1 + (H_{0n} - h_{2n}^s) T_1],$$

is a lower bound on the average costs of all feasible stationary policies in the system. From Theorem 1,

$$B_n^*/B_n^s \geq 1/\sqrt{2}. \quad (9)$$

Lemma 4. $B^s = \sum_{n=1}^N B_n^s$.

Proof. First solve the minimization problem that defines B_n^s for $n = 1, \dots, N$. Here it is helpful to distinguish among three cases: $n \in L^s$, $n \in E^s$, and $n \in G^s$. Then verify that $\sum_{n=1}^N B_n^s$ is equal to the right side of (7). \square

Lemma 5. $B^* \geq \sum_{n=1}^N B_n^*$.

Proof. From Lemma 2, $\alpha_n h_{2n} \geq \alpha_n^s h_{2n}^s$. Since $\alpha_n^s h_{2n}^s + \beta_n^s h_{1n}^s = H_{0n} = \alpha_n h_{2n} + \beta_n h_{1n}$

$$c_n(T, T_n) = K_n/T_n + \alpha_n h_{2n}(T \vee T_n) + \beta_n h_{1n} T_n \\ \geq K_n/T_n + \alpha_n^s h_{2n}^s(T \vee T_n) + \beta_n^s h_{1n}^s T_n.$$

Therefore,

$$B^* = \min_{T>0} \left\{ \frac{K_0}{T} + \sum_{n=1}^N \min_{T_n>0} c_n(T, T_n) \right\} \\ \geq \min_{T>0} \left\{ \sum_{n=1}^N \min_{T_n>0} \left(\frac{K_{0n}}{T} + \frac{K_n}{T_n} + \alpha_n^s h_{2n}^s(T \vee T_n) \right. \right. \\ \left. \left. + \beta_n^s h_{1n}^s T_n \right) \right\} \\ \geq \sum_{n=1}^N \min_{T>0, T_n>0} \left(\frac{K_{0n}}{T} + \frac{K_n}{T_n} + \alpha_n^s h_{2n}^s(T \vee T_n) \right. \\ \left. + \beta_n^s h_{1n}^s T_n \right) \\ = \sum_{n=1}^N \min_{T \geq T_n > 0} \left(\frac{K_{0n}}{T} + \alpha_n^s h_{2n}^s T + \frac{K_n}{T_n} + \beta_n^s h_{1n}^s T_n \right) \\ = \sum_{n=1}^N B_n^*,$$

where the first inequality also uses Lemma 3. \square

Theorem 3. $B^*/B^s \geq 1/\sqrt{2}$.

Proof. From Lemmas 4 and 5, $B^*/B^s \geq \sum_{n=1}^N B_n^*/\sum_{n=1}^N B_n^s \geq 1/\sqrt{2}$, where the last inequality follows from (9). \square

Theorem 4. $B^*/C^s \geq \ln 2 \approx 70\%$.

Proof. Follows from (6) and Theorem 3. \square

3. CONCLUDING REMARKS

This paper has shown that stationary policies are 70%-effective in two inventory networks: the multistage serial system and the one-warehouse multiretailer system. (We conjecture that the same worst-case bound holds in more general production/distribution systems.) This bound is tight in the sense that an example exists where the bound is reached. However, it is encouraging to note that the average effectiveness of stationary policies is very high. In a set of 1,000 randomly generated numerical examples, we observed that the average effectiveness was 99%, and the standard deviation was 1.5%. The numerical examples also suggest that the performance of stationary policies depends on the distribution of setup costs in the system. In particular, stationary policies are costly in systems where the setup cost decreases dramatically from an upstream stage to a downstream stage. We have also provided a simpler derivation of the existing lower bounds on the average costs of all feasible policies in the above systems.

This paper is partly motivated by our lack of understanding of many multiechelon, stochastic inventory systems. The research attention in the stochastic world has been focused on the study of various heuristic policies for

a very long time. Unfortunately, there is still very little evidence on how far these heuristic policies can be away from optimal, other than some limited observations based on numerical studies. (See Axsater 1993 and Federgruen 1993 for two recent reviews on the literature.) Notice that most of the heuristic policies proposed for multiechelon, stochastic inventory systems are stationary in one way or another. It is our hope that our worst-case analysis of stationary policies in deterministic systems can shed some light on the worst-case performance of the heuristic policies in stochastic systems. (Notice that Chen 1994b has identified a 94%-effective policy for a simple serial system with stochastic demand. Interestingly, his policy is nonstationary, just like the Atkins-Sun/Mitchell heuristics.)

APPENDIX A

NEW PROOFS OF LOWER BOUNDS

A key building block of this paper is the tight lower bounds on the average costs of all feasible policies for the serial system and the one-warehouse multiretailer system. These bounds have been established by Atkins and Sun (1995) and Mitchell (1987), respectively. Their proofs are based on sample-path arguments. Here we offer a different approach based on cost allocation. Below, we use the two-stage serial system to illustrate the basic idea. We refer the reader to Chen (1994a) for a complete illustration for the multistage serial system as well as the one-warehouse multiretailer system.

Consider the serial system with two stages. Let $I_1(t)$ be the on-hand inventory at stage 1 at time t , and $I_2(t)$ be the echelon on-hand inventory at stage 2 at time t (i.e., $I_1(t)$ plus the on-hand inventory at stage 2). Let $B(t)$ be the backorder level (at stage 1) at time t . Define $IL_i(t) = I_i(t) - B(t)$ and call it the echelon inventory level at stage i .

An optimal policy in the system must be nested: whenever stage 2 receives a shipment from the outside supplier, it sends a batch to stage 1. (For example, see, Schwarz 1973.) We consider only nested policies.

The rate at which the system-wide holding and backorder costs accrue is (the time index is suppressed for convenience)

$$r = h_2 I_2 + h_1 I_1 + pB \\ = h_2 IL_2 + h_1 IL_1^+ + (p + h_2) B \\ = h_2 (IL_2^+ - IL_2^-) + h_1 IL_1^+ + (p + h_2) IL_1^- \\ = [h_2 IL_2^+ + p IL_2^-] - (p + h_2) IL_2^- + h_1 IL_1^+ \\ + (p + h_2) IL_1^- \\ = r_2 + r_1,$$

where $r_2 = h_2 IL_2^+ + p IL_2^-$ and $r_1 = -(p + h_2) IL_2^- + h_1 IL_1^+ + (p + h_2) IL_1^-$. Clearly, r_2 is nonnegative. Since $r_1 = h_1 IL_1^+ + (p + h_2)(IL_1^- - IL_2^-)$ and $IL_2^- \geq IL_1^-$ by definition, r_1 is also nonnegative.

A cycle at stage 2 is the time interval between two consecutive deliveries to the stage. Figure 2 depicts the system

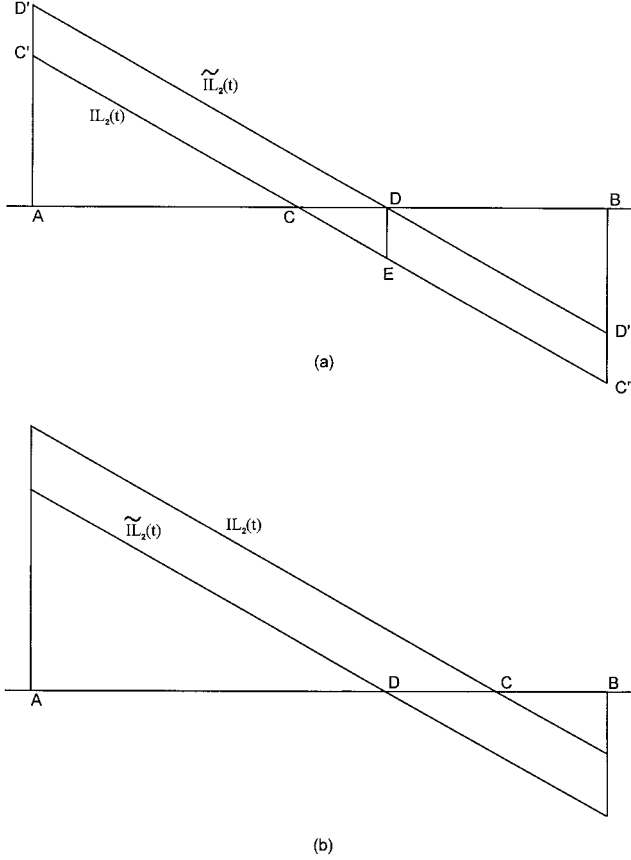


Figure 2. Fictitious and actual system inventory levels.

inventory level (i.e., $IL_2(t)$) in a cycle from time A to time B. The system inventory level decreases linearly with slope -2 over the cycle. Now consider a fictitious system inventory level in the cycle, labelled $\tilde{IL}_2(t)$, which becomes zero at time D with $\overline{AD}/\overline{AB} = p/(p + h_2)$. Note that \tilde{IL}_2 is the system inventory level that minimizes $\int_A^B r_2(t) dt$. Based on the fictitious system inventory level, define the following new cost rate

$$\begin{aligned} \tilde{r} &= [h_2 \tilde{IL}_2^+ + p \tilde{IL}_2^-] \\ &\quad + [h_1 IL_1^+ + (p + h_2) IL_1^-] 1(\tilde{IL}_2 \geq 0) \\ &= \tilde{r}_2 + \tilde{r}_1, \end{aligned}$$

where $\tilde{r}_2 = h_2 \tilde{IL}_2^+ + p \tilde{IL}_2^-$, $\tilde{r}_1 = [h_1 IL_1^+ + (p + h_2) IL_1^-] 1(\tilde{IL}_2 \geq 0)$, and $1(\cdot)$ is the indicator function.

Lemma 6. *The cost rate \tilde{r} does not overcharge holding and backorder costs in the cycle.*

Proof. First consider the case depicted in Figure 2(a), i.e., $IL_2(t) \leq \tilde{IL}_2(t)$. Suppose $IL_2(t)$ becomes zero at time C. (The case with $IL_2(t) < 0$ for all t in the cycle can be shown similarly.) Let $\Delta(A_1 \cdots A_m)$ be the area of the polygon $A_1 \cdots A_m$. Note that:

$$\begin{aligned} &\int_A^B [r_2(t) - \tilde{r}_2(t)] dt \\ &= -h_2 \Delta(C' D' DC) + p \Delta(CDD' C'') \\ &= -h_2 [\Delta(C' D' DE) - \Delta(CDE)] \\ &\quad + p [\Delta(EDD' C'') + \Delta(CDE)]. \end{aligned}$$

Since $\overline{AD}/\overline{AB} = p/(p + h_2)$, we have $h_2 \overline{AD} = p \overline{DB}$ or $h_2 \Delta(C' D' DE) = p \Delta(EDD' C'')$. Therefore,

$$\int_A^B [r_2(t) - \tilde{r}_2(t)] dt = (p + h_2) \Delta(CDE). \quad (10)$$

Since $r_1(t) - \tilde{r}_1(t) = 0$ in (A, C) , $r_1(t) - \tilde{r}_1(t) = -(p + h_2) IL_2^-(t)$ in (C, D) , and $r_1(t) = \tilde{r}_1(t) = r_1(t) \geq 0$ in (D, B) , we have

$$\begin{aligned} &\int_A^B [r_1(t) - \tilde{r}_1(t)] dt \\ &\geq -(p + h_2) \int_C^D IL_2^-(t) dt \\ &= -(p + h_2) \Delta(CDE). \end{aligned} \quad (11)$$

The lemma follows by combining (10) and (11).

Now consider the case represented in Figure 2(b). Suppose that $IL_2(t)$ becomes zero at time C. (The case with $IL_2(t) > 0$ for all t in the cycle can be handled similarly.)

Since $\tilde{IL}_2(t)$ minimizes $\int_A^B r_2(t) dt$, we have

$$\int_A^B r_2(t) dt \geq \int_A^B \tilde{r}_2(t) dt.$$

Since $r_1(t) - \tilde{r}_1(t) = 0$ in (A, D) and $r_1(t) = \tilde{r}_1(t) = r_1(t) \geq 0$ in (D, B) , we have $r_1(t) \geq \tilde{r}_1(t)$ for the entire cycle. The lemma follows. \square

Now allocate K_2 and \tilde{r}_2 to stage 2, and allocate K_1 and \tilde{r}_1 to stage 1. From Lemma 6, this cost allocation does not overcharge costs. Let us go back to Figure 2. Let t_2 be the cycle length at stage 2. Under the cost allocation, the average costs at stage 2 in $[A, B)$ are

$$K_2/t_2 + \alpha_2 h_2 t_2. \quad (12)$$

Now consider the average costs at stage 1 in $[A, D)$. Since $\tilde{IL}_2(t) > 0$ in $[A, D)$, stage 1 is essentially a single-stage EOQ system with setup cost K_1 , holding cost rate h_1 , and backorder cost rate $p + h_2$. From the EOQ model with backlogging, the average costs in an order interval of length t_1 are at least

$$K_1/t_1 + \alpha_1 h_1 t_1.$$

(An order interval is a time interval in which only one order is placed and the order occurs at the beginning of the interval.) Since the length of $[A, D)$ is $\alpha_2 t_2$ and stage 1 places an order at time A (nestedness), the length of any order interval at stage 1 in $[A, D)$ cannot exceed $\alpha_2 t_2$. Consequently, the average costs at stage 1 in $[A, D)$ are greater than or equal to

$$\min_{t_1 \leq \alpha_2 t_2} (K_1/t_1 + \alpha_1 h_1 t_1).$$

Ignoring any costs at stage 1 in $[D, B]$, we have a lower bound on the average costs at stage 1 in $[A, B]$:

$$\alpha_2 \min_{t_1 \leq \alpha_2 t_2} (K_1/t_1 + \alpha_1 h_1 t_1). \quad (13)$$

From (12) and (13), we have a lower bound on the average costs of all feasible policies in the two-stage, serial system:

$$B^* = \min [(K_2/t_2 + \alpha_2 h_2 t_2) + \alpha_2 (K_1/t_1 + \alpha_1 h_1 t_1)],$$

$$\text{s.t. } t_1 \leq \alpha_2 t_2,$$

$$t_1 > 0, t_2 > 0.$$

The form of the above lower bound can be directly generalized to the N -stage system. Define

$$B^* = \min B_N(t_N, \dots, t_1),$$

$$\text{s.t. } t_{i-1} \leq \alpha_i t_i, \quad i = 2, \dots, N,$$

$$t_i > 0, \quad i = 1, \dots, N,$$

where the function $B_N(t_N, \dots, t_1)$ is determined recursively:

$$B_n(t_n, \dots, t_1) = K_n/t_n + \alpha_n h_n t_n$$

$$+ \alpha_n B_{n-1}(t_{n-1}, \dots, t_1), \quad n = 2, \dots, N,$$

with $B_1(t_1) = K_1/t_1 + \alpha_1 h_1 t_1$. Indeed, B^* is a lower bound on the average costs of all feasible policies in the N -stage system. A simple change of variables leads to the form given in subsection 1.4.

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