

ONE-WAREHOUSE MULTIRETAILER SYSTEMS WITH CENTRALIZED STOCK INFORMATION

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We consider a distribution system with a central warehouse and multiple retailers. The warehouse orders from an outside supplier and replenishes the retailers which in turn satisfy customer demand. The retailers are nonidentical, and their demand processes are independent compound Poisson. There are economies of scale in inventory replenishment, which is controlled by an echelon-stock, batch-transfer policy. For the special case with simple Poisson demand, we develop an exact method for computing the long-run average holding and backorder costs of the system. Based on this exact method, we provide approximations for compound Poisson demand. Numerical examples are used to illustrate the accuracy of the approximations. We also present a numerical comparison between the average costs of a heuristic, echelon-stock policy and an existing lower bound on the average costs of all feasible policies.

More and more companies now possess centralized stock information, due to modern information technologies. For example, Wal-Mart, a discount retailer, has a satellite communication system that transmits daily point-of-sale data to its distribution centers and suppliers (Stalk et al. 1992). In this paper, we study a class of replenishment policies that use centralized stock information.

Centralized stock information can be utilized through echelon-stock policies. A facility's echelon stock is its on-hand inventory plus the inventories at all its downstream facilities. A replenishment policy based on echelon-stock information is called an echelon-stock policy. Although the concept of echelon stock was introduced by Clark and Scarf (1960) some three decades ago, most research has focused on installation-stock policies that use only local stock information. The proliferation of information technologies and the resulting availability of centralized stock information bring back the interest in echelon-stock policies.

We consider a distribution system where a warehouse orders from an outside supplier and replenishes N retailers. The retailers are nonidentical and face independent compound Poisson demands. All excess demands at the retailers are completely backlogged. Inventory transfers from the warehouse to the retailers incur fixed costs. Inventory transfers between retailers are not allowed. We propose a class of echelon-stock, batch-transfer policies for the system: each facility uses an echelon-stock (R, nQ) policy, i.e., once its echelon stock falls to or below the reorder point R , it orders an integer multiple of Q —its base order quantity—to raise its echelon stock to above R . When the demand is simple Poisson, an (R, nQ) policy reduces to the well known (R, Q) policy where each order is of exactly Q units. As the reader may have noticed, the

above class of policies is an extension of those studied by De Bodt and Graves (1985) and Chen and Zheng (1994a) in serial systems.

For the special case with simple Poisson demand, we provide an exact method for computing the long-run average holding and backorder costs of the system. This is achieved by disaggregating the backorders at the warehouse among the retailers. Although the disaggregation method fails to generate an exact evaluation scheme for compound Poisson demand, it can be used to develop approximations. Numerical examples are used to illustrate the accuracy of the approximations. We also present a numerical comparison between the average costs of a heuristic, echelon-stock (R, nQ) policy with an existing lower bound on the average costs of all feasible policies. The gap between the two is within a reasonable range, suggesting that the heuristic policy is not far from optimal.

One-warehouse multi-retailer systems have received much research attention. Base stock policies have been extensively studied for systems without setup costs; see Axsater (1993b) and Federgruen (1993) for comprehensive reviews. For systems with setup costs, most of the previous work has been confined to continuous-review, installation-stock (R, Q) policies (Deuermeyer and Schwarz 1981, Moinzadeh and Lee 1986, Lee and Moinzadeh 1987a and b, Svoronos and Zipkin 1988, and Axsater 1993a). Under an installation-stock (R, Q) policy, the warehouse orders according to its local stock level, treating orders from the retailers as its demand. Since the retailers order in batches, the demand the warehouse faces is an aggregated version of, thus more volatile than, the customer demand process. In contrast, echelon-stock policies react directly to customer demand.

Subject classifications: Inventory/production: echelon-stock, batch-transfer policies. Multiechelon: nonidentical retailers. Inventory/production, stochastic: compound Poisson demand, continuous review.

Area of review: STOCHASTIC PROCESSES AND THEIR APPLICATIONS.

Evaluating the performance of control policies in one-warehouse multi-retailer systems with stochastic demand has been a major challenge. So far, there are three distinctive approaches for developing an evaluation procedure. One approach is by characterizing the stochastic leadtime of a retailer order, which consists of the transportation time from the warehouse to the retailer and a random delay, or retard, at the warehouse. This method underscores most of the existing evaluation schemes. Another approach is to match every supply unit with a demand unit. By keeping track of an arbitrary supply unit—when it enters the system, how it moves through the system, and when it exits—one is able to calculate the holding and backorder costs associated with this unit. These costs are then converted to long-run average costs. The idea of matching supply units with demand first appeared in Svoronos and Zipkin, and was later developed into an evaluation method by Axsater (1990, 1993a). Recently, Axsater (1993c) applied this method to the class of echelon-stock policies proposed here and developed an alternative evaluation scheme. The third approach is to derive the steady state distributions of inventory levels by disaggregating the backorders at the warehouse among the retailers (Graves 1985). Our method falls into the last category.

The rest of the paper consists of five sections and three appendices. Section 1 specifies the model, introduces notation, and presents a preliminary observation. Section 2 develops an exact and an approximate evaluation procedure for systems with simple Poisson demand. Section 3 provides two approximations for systems with compound Poisson demand. Section 4 reports numerical examples. Section 5 concludes the paper. Appendix I contains the proof of a proposition, Appendix II discusses computational issues associated with the exact as well as the approximate procedures, and Appendix III explains why the exact method for the simple Poisson case fails to carry over to the compound Poisson case.

1. MODEL, NOTATION AND PRELIMINARIES

Consider a one-warehouse N -retailer system where the retailers order from the warehouse, which in turn orders from an outside supplier with unlimited stock. For convenience, we will also refer to the warehouse as facility 0 and retailer i as facility i for $i = 1, \dots, N$. Each order incurs a fixed setup cost. Transportation times are constant: L_0 is the transportation time from the outside supplier to the warehouse, and L_i is the transportation time from the warehouse to retailer i . Let $h_i \geq 0$ be the unit echelon holding cost rate at facility $i = 0, \dots, N$. The retailers face independent compound Poisson demand. The demand process at retailer i can be described as follows: customers arrive according to a Poisson process with rate λ_i and the demand sizes of different customers are independent and identically distributed. Let D_i be the demand size of a customer at retailer i . We assume that D_i is integral and

$\Pr(D_i = 0) = 0$ and $\Pr(D_i = 1) > 0$ for $i = 1, \dots, N$. Excess demand at the retailers is completely backordered, and the unit backorder cost rate is p_i at retailer i .

For any time t , define

$NIP_0(t)$ = nominal echelon inventory position or echelon stock of the warehouse
= inventories in transit to or at all facilities minus customer backorders at the retailers.

$NIP_i(t)$ = nominal echelon inventory position or echelon stock of retailer i
= retailer i 's on-hand inventory plus outstanding orders (either in transit or backlogged at the warehouse) minus customer backorders at the retailer.

$IP_i(t)$ = echelon inventory position of retailer i
= $NIP_i(t)$ minus its outstanding orders that are backlogged at the warehouse.

$IL_i(t)$ = echelon inventory level at facility i
= $NIP_i(t)$ minus the facility's outstanding orders.

$B^i(t)$ = number of base-lots of retailer i 's orders backlogged at the warehouse.
(The base-lot is defined in the next paragraph.)

$B_0(t)$ = number of base-lots backordered at the warehouse = $\sum_{i=1}^N B^i(t)$.

$B_i(t)$ = customer backorders at retailer i .

$I_i(t)$ = echelon inventory at facility i
= $IL_i(t) + B_i(t)$ for retailer i
= $IL_0(t) + \sum_{i=1}^N B_i(t)$ for the warehouse.

$D_i(t_1, t_2)$ = total demand at retailer i in the time interval (t_1, t_2) .

$D_0(t_1, t_2)$ = total system demand in the time interval (t_1, t_2) = $\sum_{i=1}^N D_i(t_1, t_2)$.

When the time index t is omitted, the notation represents steady state variables. All variables are integral.

Facility i , $i = 0, \dots, N$, uses an echelon-stock (R_i, nQ_i) policy: it orders nQ_i units from its supplier once its echelon stock y falls to or below its reorder point R_i , where n is the smallest integer with $y + nQ_i > R_i$. If the on-hand inventory at the warehouse is insufficient to satisfy a retailer order, the order is *partially filled* with the remainder backordered at the warehouse. Backlogged retailer orders are satisfied on a *first-come, first-served* basis. We assume *integer-ratio base order quantities*: $Q_i = m_i Q_N$ for $i = 0, \dots, N - 1$ where m_i is a positive integer. We will refer to Q_N as the *base-lot* of the system. Since inventory costs tend to be insensitive to the choice of order quantities (Zheng 1992 and Zheng and Chen 1992), we believe that such a restriction should not be too costly. More important, integer-ratio order quantities facilitate quantity coordination among different facilities, simplifying packaging, transportation and stock counts. Many companies have recognized these managerial benefits of integer-ratio policies, e.g., Campbell Soup Company requires its retailers to

order integer multiples of 1/4 of a pallet for any of its canned products.

Finally, note that $IL_0(t) - \sum_{i=1}^N NIP_i(t)$ is the inventory level at the warehouse: its on-hand inventory minus backlogged retailer orders. We assume that the initial on-hand inventory at the warehouse is an integer multiple of the base-lot. (This can always be achieved by initially sending the residual units, if any, at the warehouse to the retailers.) Since every shipment to or from the warehouse is an integer multiple of the base-lot, we have

$$IL_0(t) - \sum_{i=1}^N NIP_i(t) = mQ_N, \quad m \text{ integer.} \quad (1)$$

If $m < 0$ then $B_0(t) = -m$.

2. SIMPLE POISSON DEMAND

In this section, we assume that the demand processes are simple Poisson, i.e., $\Pr(D_i = 1) = 1$ for $i = 1, \dots, N$. We develop an exact and an approximate procedure to evaluate the long-run average holding and backorder costs of any feasible echelon-stock (R, Q) policy. (Note that (R, nQ) policies reduce to (R, Q) policies under simple Poisson demand.)

2.1. Exact Evaluation

The rate at which the system-wide holding and backorder costs accrue at time t is

$$\sum_{i=0}^N h_i I_i(t) + \sum_{i=1}^N p_i B_i(t),$$

which is equal to

$$\sum_{i=0}^N h_i IL_i(t) + \sum_{i=1}^N (p_i + h_i + h_0) B_i(t).$$

Note that $B_i(t) = (IL_i(t))^-$ for $i = 1, \dots, N$, where $x^- = \max\{-x, 0\}$. To obtain the long-run average holding and backorder costs, we need only to derive the distributions of IL_i for $i = 0, \dots, N$.

The distribution of IL_0 is rather easy to obtain. Note that

$$IL_0(t + L_0) = NIP_0(t) - D_0(t, t + L_0), \quad (2)$$

and that the steady state distribution of $NIP_0(t)$ is uniform over $R_0 + 1, \dots, R_0 + Q_0$ (Hadley and Whitin 1963). Therefore, the distribution of IL_0 can be obtained through a simple convolution by (2). On the other hand, we have for retailer i

$$IL_i(t + L_i) = IP_i(t) - D_i(t, t + L_i). \quad (3)$$

Therefore, the distribution of IL_i can be obtained from the distribution of IP_i . The task for the rest of this subsection is to derive the distribution of IP_i .

Define $Z_i(t) = NIP_i(t) - R_i$ for $i = 0, \dots, N$. Thus $1 \leq Z_i(t) \leq Q_i$. Define $\mathbf{Z}(t) = (Z_1(t), \dots, Z_N(t))$, $\mathbf{z} = (z_1, \dots, z_N)$, and $|\mathbf{z}| = \sum_{i=1}^N z_i$.

Proposition 1. $IL_0, Z_1, \dots, Z_{N-1}$ are independent.

Proof. See Appendix I. \square

From (1), we have

$$Z_N(t) = IL_0(t) - \sum_{i=1}^{N-1} Z_i(t) - \sum_{i=1}^N R_i - mQ_N,$$

where m is an integer. Since $1 \leq Z_N(t) \leq Q_N$, it is easy to see that the values of $IL_0(t), Z_0(t), \dots, Z_{N-1}(t)$ uniquely determine the value of $Z_N(t)$. Note that each Z_i is uniformly distributed from 1 to Q_i . Let $g(\cdot)$ be the density function of IL_0 . From Proposition 1, we have

$$\begin{aligned} \Pr(IL_0 = w, \mathbf{Z} = \mathbf{z}) &= \Pr(IL_0 = w, Z_i = z_i, i = 1, \dots, N-1) \\ &= g(w) \prod_{i=1}^{N-1} Q_i, \end{aligned} \quad (4)$$

if $1 \leq z_i \leq Q_i$ for $i = 1, \dots, N$ and $w = |\mathbf{z}| + \sum_{i=1}^N R_i + mQ_N$ for some integer m , and $\Pr(IL_0 = w, \mathbf{Z} = \mathbf{z}) = 0$ otherwise.

Proposition 2. For any $b \geq 0$ and any \mathbf{z} with $1 \leq z_i \leq Q_i$ for $i = 1, \dots, N$, we have

$$\begin{aligned} \pi(b, \mathbf{z}) &\stackrel{\text{def}}{=} \Pr(B_0 = b, \mathbf{Z} = \mathbf{z}) \\ &= \begin{cases} g\left(-bQ_N + \sum_{i=1}^N R_i + |\mathbf{z}|\right) \prod_{i=1}^{N-1} Q_i, & b > 0, \\ \sum_{m \geq 0} g\left(mQ_N + \sum_{i=1}^N R_i + |\mathbf{z}|\right) \prod_{i=1}^{N-1} Q_i, & b = 0. \end{cases} \end{aligned}$$

Proof. From (1), we have

$$\left(\sum_{i=1}^N NIP_i - IL_0\right)^+ = B_0 Q_N,$$

where $x^+ = \max\{x, 0\}$. Therefore, the event $(B_0 = b, \mathbf{Z} = \mathbf{z})$ is equivalent to the event $(IL_0 = -bQ_N + \sum_{i=1}^N R_i + |\mathbf{z}|, \mathbf{Z} = \mathbf{z})$ if $b > 0$ and to the event $(IL_0 = mQ_N + \sum_{i=1}^N R_i + |\mathbf{z}|, \mathbf{Z} = \mathbf{z}, m = 0, 1, \dots)$ if $b = 0$. The proposition follows from (4). Since $IL_0 \leq R_0 + Q_0$, the sum over m involves only a finite number of terms. \square

Define $\mathcal{N} = \{1, \dots, N\}$. For any subset S of \mathcal{N} , let $n(S, y)$ be the number of different vectors in the set

$$\left\{ (z_i)_{i \in S} : 1 \leq z_i \leq Q_i, i \in S, \text{ and } \sum_{i \in S} z_i = y \right\}.$$

For any $i \in \mathcal{N}$, note that $n(\{i\}, y) = 1$ for $y = 1, \dots, Q_i$ and $n(\{i\}, y) = 0$ otherwise. Furthermore, we have

$$n(S \cup \{i\}, y) = \sum_{z=1}^{Q_i} n(S, y - z), \quad (5)$$

for any subset S of \mathcal{N} and any $i \in \mathcal{N} \setminus S$. Letting $S = \mathcal{N} \setminus \{i\}$ in (5), it is easy to verify that

$$\begin{aligned} n(\mathcal{N} \setminus \{i\}, y) &= n(\mathcal{N}, y + 1) - n(\mathcal{N}, y) \\ &\quad + n(\mathcal{N} \setminus \{i\}, y - Q_i). \end{aligned} \quad (6)$$

Therefore, we can use (5) recursively to obtain $n(\mathcal{N}, y)$ and use (6) for $n(\mathcal{N} \setminus \{i\}, y)$ for all $i \in \mathcal{N}$.

For any $i \in \mathcal{N}$, define $\mathbf{Z}_{-i} = (Z_k, k \in \mathcal{N} \setminus \{i\})$, $\mathbf{z}_{-i} = (z_k, k \in \mathcal{N} \setminus \{i\})$ and $Y_{-i} = |Z_{-i}| = \sum_{k \neq i} Z_k$. Let

$$\pi_i(b, z, y) = \Pr(B_0 = b, Z_i = z, Y_{-i} = y).$$

Thus

$$\pi_i(b, z, y) = \sum_{\mathbf{z}: z_i = z, |Z_{-i}| = y} \pi(b, \mathbf{z}).$$

From Proposition 2, we have for any $b > 0$,

$$\pi_i(b, z, y) = n(\mathcal{N} \setminus \{i\}, y) g\left(-bQ_N + \sum_{k=1}^N R_k + z + y\right) / \prod_{k=1}^{N-1} Q_k, \quad (7)$$

and for $b = 0$

$$\pi_i(0, z, y) = n(\mathcal{N} \setminus \{i\}, y) \sum_{m \geq 0} g\left(mQ_N + \sum_{k=1}^N R_k + z + y\right) / \prod_{k=1}^{N-1} Q_k. \quad (8)$$

Let t be a time epoch at steady state. Take any $i \in \mathcal{N}$. Suppose $B_0(t) = b$, $Z_i(t) = z$ and $Y_{-i}(t) = y$. (The time index will be omitted for convenience.) We proceed to disaggregate the b base-lots of warehouse backorders between retailer i and the group $\mathcal{N} \setminus \{i\}$. This disaggregation depends on the ordering processes of the retailer and the group *before* time t . And the ordering process of the group $\mathcal{N} \setminus \{i\}$ is determined by its demand process before time t and \mathbf{Z}_{-i} . Since

$$\Pr(\mathbf{Z}_{-i} = \mathbf{z}_{-i} | B_0 = b, Z_i = z, Y_{-i} = y) = \begin{cases} 1/n(\mathcal{N} \setminus \{i\}, y) & \text{if } |\mathbf{z}_{-i}| = y, \\ 0 & \text{otherwise,} \end{cases}$$

we know that, given $Y_{-i} = y$, the distribution of \mathbf{Z}_{-i} , and thus the ordering process of the group $\mathcal{N} \setminus \{i\}$, is independent of the values of B_0 and Z_i . Now we are ready to introduce the following definitions. For convenience, we label demands and orders before time t backward. Therefore, e.g., the “first” demand at a retailer refers to the “last” demand at the retailer before time t and the “first” order placed by a retailer means the “last” order by the retailer before time t . Define

$P_{-i}^y(n|j)$ = probability that the group of retailers, $\mathcal{N} \setminus \{i\}$, has placed *at most* n base-lots of orders by the time the j th demand arrives in the group (or by the j th group demand) given $|Z_{-i}| = y$.

Below, we develop a recursive procedure to compute $P_{-i}^y(n|j)$. To that end, we need the following additional definitions. For any retailer k and any subset S of \mathcal{N} , define

$p_k^y(n|j)$ = probability that retailer k has placed *exactly* n base-lots of orders by its j th demand given $Z_k = y$,

$P_S^y(n|j)$ = probability that the group of retailers, S , has placed *at most* n base-lots of orders by the j th group demand given $\sum_{k \in S} Z_k = y$.

Note that $P_{-i}^y(n|j) = P_{\mathcal{N} \setminus \{i\}}^y(n|j)$.

Consider $p_k^y(n|j)$. Note that $p_k^y(0|0) = 1$ and $p_k^y(n|0) = 0$ for all $n \geq 1$. Recall that $Q_k = m_k Q_N$ where m_k is a positive integer. Since each order placed by retailer k is exactly Q_k units, or m_k base-lots, $p_k^y(n|j) = 0$ if n is not an integer multiple of m_k . Now assume $n = lm_k$ for some nonnegative integer l . Note that the first order at retailer k is triggered by the $(Q_k + 1 - y)$ th demand, the second order by the $[(Q_k + 1 - y) + Q_k]$ th demand, and so on. Consequently, we have

$$p_k^y(lm_k|j) = \begin{cases} 1 & \text{if } lQ_k + 1 - y \leq j < (l+1)Q_k + 1 - y, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Note that the above probabilities reflect a deterministic situation and do not depend on customer arrival times. By definition,

$$P_{\{k\}}^y(n|j) = \sum_{m=0}^n p_k^y(m|j). \quad (10)$$

Now consider $P_{S \cup \{k\}}^y(n|j)$ where $k \in \mathcal{N} \setminus S$. Let $Y_S = \sum_{l \in S} Z_l$. Note that

$$\Pr(Y_S = y - z, Z_k = z | Y_S + Z_k = y) = \begin{cases} n(S, y - z) / n(S \cup \{k\}, y) & \text{if } 1 \leq z \leq Q_k, \\ 0 & \text{otherwise.} \end{cases}$$

Of the first j demands of the group $S \cup \{k\}$, let l be the number of demands of retailer k . Clearly, l has a binomial distribution. Define

$$b(\alpha, \beta; l, j) = \binom{j}{l} \left(\frac{\alpha}{\beta}\right)^l \left(1 - \frac{\alpha}{\beta}\right)^{j-l}.$$

Therefore,

$$P_{S \cup \{k\}}^y(n|j) = \sum_{z=1}^{Q_k} \frac{n(S, y - z)}{n(S \cup \{k\}, y)} \sum_{l=0}^j b(\lambda_k, \lambda_k + \lambda_S; l, j) \times \sum_{m=0}^n p_k^z(m|l) P_S^{y-z}(n - m | j - l) \quad (11)$$

where $\lambda_S = \sum_{l \in S} \lambda_l$. By using (11) recursively, one can obtain $P_{-i}^y(n|j)$ for any $i \in \mathcal{N}$. See Appendix II.1 for an alternative procedure.

For $b_i \geq 0$, define

$$P_i^{b_i, z, y}(b_i) = \Pr(B^i \geq b_i | B_0 = b, Z_i = z, Y_{-i} = y).$$

Let $m = \lceil b_i Q_N / Q_i \rceil$ where $\lceil x \rceil$ is the minimum integer that is greater than or equal to x . Thus the b_i th base-lot of retailer i is included in its m th order (before time t). Recall that the m th order of retailer i is triggered by the $J_i(z, b_i)$ th demand of the retailer where

$$J_i(z, b_i) = mQ_i + 1 - z.$$

Since the warehouse has a total of b base-lots of backlogged retailer orders at time t and retailer orders are satisfied at the warehouse on a first-come, first-served basis, $P_i^{b_i, z, y}(b_i)$ is equal to the probability that by the $J_i(z,$

b_i)th demand at retailer i , the total quantity (in base-lots) ordered by all the other retailers is *at most* $b - b_i$. Notice that by the $J_i(z, b_i)$ th demand at retailer i , the number of demands in the group $\mathcal{N} \setminus \{i\}$, denoted by j , follows a negative binomial distribution. Define

$$b_i^n(m, j) = \binom{j+m-1}{j} \left(\frac{\lambda_i}{\lambda_0}\right)^m \left(1 - \frac{\lambda_i}{\lambda_0}\right)^j, \quad j = 0, 1, \dots$$

where $\lambda_0 = \sum_{k=1}^N \lambda_k$. Consequently,

$$P_i^{b,z,y}(b_i) = \sum_{j=0}^{\infty} b_i^n(J_i(z, b_i), j) P_{-i}^y(b - b_i | j) \quad \text{for } b_i > 0, \quad (12)$$

and $P_i^{b,z,y}(0) = 1$. Let $\hat{Z}_i = IP_i - R_i$. By definition, we have $IP_i = NIP_i - B^i Q_N$ or $\hat{Z}_i = Z_i - B^i Q_N$. Therefore,

$$\Pr(\hat{Z}_i \leq w) = \sum_{(b,z,y)} P_i^{b,z,y}((z-w)/Q_N) \pi_i(b, z, y). \quad (13)$$

The distribution of IP_i can be obtained from $\Pr(IP_i \leq w) = \Pr(\hat{Z}_i \leq w - R_i)$, and the distribution of IL_i from (3). The development for the exact evaluation procedure is now complete.

We end this subsection with an observation on the system-wide holding and backorder cost function. Suppose that the base order quantities are all given. Define $r_0 = R_0 - \sum_{i=1}^N R_i$. Let $C(r_0, R_1, \dots, R_N)$ be the average total holding and backorder costs, i.e.,

$$C(r_0, R_1, \dots, R_N) = E[h_0 IL_0] + \sum_{i=1}^N E[h_i IL_i + (p_i + h_i + h_0) B_i].$$

Note that

$$E[IL_0] = R_0 + \frac{Q_0 + 1}{2} - \lambda_0 L_0.$$

Define

$$G_i(y) = E[h_i(y - D_i(t, t + L_i)) + (p_i + h_i + h_0)(y - D_i(t, t + L_i))^-].$$

Thus $G_i(\cdot)$ is convex. Therefore,

$$C(r_0, R_1, \dots, R_N) = h_0 r_0 + \sum_{i=1}^N [h_0 R_i + E G_i(\hat{Z}_i + R_i)] + \frac{(Q_0 + 1)h_0}{2} - \lambda_0 L_0 h_0.$$

Now fix r_0 . Let $g_0(w) = \Pr(Z_0(t) - D_0(t, t + L_0) = w)$ at steady state. Thus $g(w) = g_0(w - R_0)$. Note that $g_0(\cdot)$ is independent of R_k for $k = 0, 1, \dots, N$. From (7) and (8), we have for any $b > 0$,

$$\pi_i(b, z, y) = n(\mathcal{N} \setminus \{i\}, y) g_0(-b Q_N - r_0 + z + y) \prod_{k=1}^{N-1} Q_k,$$

and for $b = 0$

$$\pi_i(0, z, y) = n(\mathcal{N} \setminus \{i\}, y) \sum_{m \geq 0} g_0(m Q_N - r_0 + z + y) \prod_{k=1}^{N-1} Q_k.$$

Consequently, $\pi_i(b, z, y)$ is independent of R_k for $k = 1, \dots, N$. On the other hand, recall that $P_i^{b,z,y}(b_i)$ depends on the ordering processes at the retailers, which are also independent of R_k for $k = 1, \dots, N$. As a result, the distribution of \hat{Z}_i is independent of R_k for $k = 1, \dots, N$. Therefore, $E G_i(\hat{Z}_i + R_i)$ is convex in R_i .

Proposition 3. For any fixed $r_0, C(r_0, R_1, \dots, R_N)$ is separable and convex in $R_i, i = 1, \dots, N$.

2.2. Approximation

Let t be a time epoch at steady state. Label demands and orders before time t backward. Let k and i be elements of \mathcal{N} , and S a subset of \mathcal{N} . Define

$p_k(n|j)$ = probability that retailer k has placed *exactly* n base-lots of orders by its j th demand (before time t),

$P_S(n|j)$ = probability that the group of retailers, S , has placed *at most* n base-lots of orders by the j th group demand,

$$P_{-i}(n|j) = P_{\mathcal{N} \setminus \{i\}}(n|j).$$

First consider $p_k(n|j)$. Clearly, $p_k(n|j) = 0$ if n is not an integer multiple of m_k . From (9), we have

$$p_k(lm_k|j) = \Pr(lQ_k + 1 - Z_k(t) \leq j < (l+1)Q_k + 1 - Z_k(t)),$$

where l is a nonnegative integer. (We will omit the time index t for convenience.) Since Z_k is uniformly distributed over $1, 2, \dots, Q_k$, we have

$$p_k(lm_k|j) = \begin{cases} j/Q_k - l + 1, & (l-1)Q_k + 1 \leq j < lQ_k, \\ l + 1 - j/Q_k, & lQ_k \leq j \leq (l+1)Q_k - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

By definition, we have

$$P_{\{k\}}(n|j) = \sum_{m=0}^n p_k(m|j). \quad (15)$$

Now consider $P_{S \cup \{k\}}(n|j)$ where $k \in \mathcal{N} \setminus S$. Similar to (11), we have

$$P_{S \cup \{k\}}(n|j) = \sum_{l=0}^j b(\lambda_k, \lambda_k + \lambda_S; l, j) \sum_{m=0}^n p_k(m|l) \times P_S(n-m|j-l). \quad (16)$$

We can use (16) recursively to obtain $P_{-i}(n|j)$ for all $i \in \mathcal{N}$. Appendix II.1 provides an alternative procedure.

Take any $i \in \mathcal{N}$. Assume that (B_0, Z_i) is independent of \mathbf{Z}_{-i} . From Proposition 2, this is clearly an approximation. Define

$$P_i^{b,z}(b_i) = \Pr(B^i \geq b_i | B_0 = b, Z_i = z).$$

Therefore,

$$P_i^{b,z}(b_i) = \sum_{j=0}^{\infty} b_i^n (J_i(z, b_i), j) P_{-i}(b - b_i | j), \quad b_i > 0, \quad (17)$$

and $P_i^{b,z}(0) = 1$. Let

$$\pi_i(b, z) \stackrel{\text{def}}{=} \Pr(B_0 = b, Z_i = z) = \sum_y \pi_i(b, z, y), \quad (18)$$

where the summation over y is from $N - 1$ to $\sum_{k \neq i} Q_k$. Similar to (13), we have

$$\Pr(\hat{Z}_i \leq w) = \sum_{(b,z)} P_i^{b,z}((z - w)/Q_N) \pi_i(b, z). \quad (19)$$

(For systems with identical retailers, the approximation can be further simplified by assuming $P_{-i}(n | j) = 1$ if $j = nQ_N$ and $P_{-i}(n | j) = 0$ otherwise. This approximation has been used in evaluating installation-stock (R, Q) policies.)

3. COMPOUND POISSON DEMAND

In this section, we develop two approximations for systems with compound Poisson demand, i.e., $\Pr(D_i = 1) < 1$ for some retailer i . The first approximation is directly adapted from the exact method for the simple Poisson case, while the second one is essentially the same as the approximation for the simple Poisson case.

3.1. Approximation I

Propositions 1 and 2 are both valid here. Consequently, we have the steady state joint distribution of the warehouse backorders (B_0) and the (relative) nominal inventory positions of the retailers (Z) .

Consider any retailer i . Let t be a time epoch at steady state. Suppose $B_0(t) = b$, $Z_i(t) = z$ and $Y_{-i}(t) = y$. Assume that the demand processes at the retailers before time t can be characterized by the original independent compound Poisson processes. (This is an approximation since the warehouse backorder level contains some relevant information about the demand processes before time t . See Appendix III for more details.) Under this assumption, we proceed to disaggregate the warehouse backorders between retailer i and the group $\mathcal{N} \setminus \{i\}$.

Label customers, demand units and orders before time t backward. Parallel to the simple Poisson case, we have the following definitions:

- $p_k^y(n | [j])$ = probability that retailer k has placed *exactly* n base-lots of orders by its j th customer (before time t) given $Z_k = y$,
- $P_S^y(n | [j])$ = probability that the group of retailers, S , has placed *at most* n base-lots of orders by the j th customer in the group given $\sum_{k \in S} Z_k = y$,
- $P_{-i}^y(n | [j]) = P_{\mathcal{N} \setminus \{i\}}^y(n | j)$.

Note that since each customer demands a random number of units, we need to distinguish between the j th customer

and the j th demand unit. For easy distinction, we use $[j]$ to denote the j th customer.

It is clear that the expression for $p_k^y(n | j)$ in Section 2 remains valid if we interpret j as the j th demand unit at retailer k . Let $D_k[j]$ be the total demand of the first j customers at retailer k , with $D_k[0] \equiv 0$. Therefore,

$$p_k^y(n | [j]) = E p_k^y(n | D_k[j]).$$

Let $F_k^j(\cdot)$ be the cumulative distribution function of $D_k[j]$. From (9), we have for any nonnegative integer l

$$p_k^y(l m_k | [j]) = F_k^j((l + 1)Q_k - y) - F_k^j(lQ_k - y), \quad (20)$$

and $p_k^y(n | [j]) = 0$ if n is not a nonnegative integer multiple of m_k . From definition, we have

$$P_{\{k\}}^y(n | [j]) = \sum_{m=0}^n p_k^y(m | [j]). \quad (21)$$

Now consider $P_{S \cup \{k\}}^y(n | [j])$ for $k \in \mathcal{N} \setminus S$. Similar to (11), we have

$$\begin{aligned} P_{S \cup \{k\}}^y(n | [j]) &= \sum_{z=1}^{Q_k} \frac{n(S, y - z)}{n(S \cup \{k\}, y)} \sum_{l=0}^j b(\lambda_k, \lambda_k + \lambda_S; l, j) \\ &\times \sum_{m=0}^n p_k^z(m | [l]) P_S^{y-z}(n - m | [j - l]). \end{aligned} \quad (22)$$

We can use (22) recursively to compute $P_{-i}^y(n | [j])$ for every retailer i . See Appendix II.1 for an alternative procedure.

For any positive integer M , define

$$\begin{aligned} \alpha_i(m, M) &= F_i^{m-1}(M - 1) - F_i^m(M - 1), \\ & \quad m = 1, \dots, M. \end{aligned} \quad (23)$$

Note that $\alpha_i(m, M)$ is the probability that the m th customer demands the M th unit at retailer i . Similar to (12), we have

$$\begin{aligned} P_i^{b,z,y}(b_i) &= \sum_{m=1}^{J_i(z, b_i)} \alpha_i(m, J_i(z, b_i)) \sum_{j=0}^{\infty} b_i^m(m, j) \\ &\times P_{-i}^y(b - b_i | [j]), \quad b_i > 0, \end{aligned} \quad (24)$$

and $P_i^{b,z,y}(0) = 1$. The distribution of \hat{Z}_i (thus IP_i) can be obtained by (13).

3.2. Approximation II

First determine $g(\cdot)$, the steady state distribution of the system inventory level. This can be done exactly, see (2). Let $\mu_i = ED_i$. Define $\tilde{\lambda}_i = \lambda_i \mu_i$. Then assume the demand processes at the retailers are independent simple Poisson processes with rate $\tilde{\lambda}_i$ at retailer i , and follow the approximation in Subsection 2.2. This leads to $\Pr(\hat{Z}_i \leq w)$ and thus the distribution of IP_i . Finally, use the exact leadtime demand distribution in (3) to obtain the distribution of IL_i .

Table I
Examples with Simple Poisson Demand

| No. | N | λ^1 | λ^2 | λ^3 | λ^4 | Q_0 | Q^1 | Q^2 | Q^3 | Q^4 | R_0 | R^1 | R^2 | R^3 | R^4 |
|-----|---|-------------|-------------|-------------|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | | 1 | 1 | 1 | 1 | 32 | 8 | 4 | 4 | 2 | 13 | 0 | 1 | 1 | 2 |
| 2 | | 1 | 1 | 1 | 2 | 40 | 8 | 4 | 4 | 4 | 15 | 0 | 1 | 1 | 3 |
| 3 | | 1 | 1 | 2 | 1 | 40 | 8 | 4 | 4 | 2 | 15 | 0 | 1 | 3 | 1 |
| 4 | | 1 | 1 | 2 | 2 | 48 | 8 | 4 | 4 | 4 | 17 | 0 | 1 | 3 | 3 |
| 5 | | 1 | 2 | 1 | 1 | 40 | 8 | 8 | 4 | 2 | 15 | 0 | 2 | 1 | 2 |
| 6 | | 1 | 2 | 1 | 2 | 48 | 8 | 8 | 4 | 4 | 18 | 0 | 2 | 1 | 3 |
| 7 | | 1 | 2 | 2 | 1 | 48 | 8 | 8 | 4 | 2 | 17 | 0 | 2 | 3 | 2 |
| 8 | 4 | 1 | 2 | 2 | 2 | 28 | 4 | 8 | 4 | 4 | 24 | 1 | 1 | 2 | 2 |
| 9 | | 2 | 1 | 1 | 1 | 40 | 8 | 4 | 4 | 2 | 14 | 2 | 1 | 1 | 2 |
| 10 | | 2 | 1 | 1 | 2 | 48 | 8 | 4 | 4 | 4 | 17 | 2 | 1 | 1 | 3 |
| 11 | | 2 | 1 | 2 | 1 | 48 | 8 | 4 | 4 | 2 | 16 | 1 | 1 | 3 | 1 |
| 12 | | 2 | 1 | 2 | 2 | 28 | 8 | 4 | 4 | 4 | 24 | 1 | 1 | 2 | 2 |
| 13 | | 2 | 2 | 1 | 1 | 48 | 8 | 8 | 4 | 2 | 16 | 2 | 2 | 1 | 2 |
| 14 | | 2 | 2 | 1 | 2 | 28 | 8 | 8 | 4 | 4 | 24 | 1 | 1 | 1 | 3 |
| 15 | | 2 | 2 | 2 | 1 | 56 | 8 | 8 | 4 | 2 | 18 | 1 | 1 | 3 | 1 |
| 16 | | 2 | 2 | 2 | 2 | 32 | 8 | 8 | 4 | 4 | 27 | 1 | 1 | 2 | 2 |
| 17 | | 1 | 1 | 1 | 1 | 32 | 4 | 4 | 4 | 2 | 26 | 1 | 1 | 1 | 1 |
| 18 | | 1 | 1 | 1 | 2 | 40 | 4 | 4 | 4 | 4 | 30 | 1 | 1 | 1 | 3 |
| 19 | | 1 | 1 | 2 | 1 | 40 | 4 | 4 | 4 | 2 | 30 | 0 | 0 | 2 | 1 |
| 20 | | 1 | 1 | 2 | 2 | 48 | 4 | 4 | 4 | 4 | 35 | 0 | 0 | 2 | 2 |
| 21 | | 1 | 2 | 1 | 1 | 40 | 4 | 8 | 4 | 2 | 32 | 1 | 1 | 1 | 1 |
| 22 | | 1 | 2 | 1 | 2 | 48 | 4 | 8 | 4 | 4 | 36 | 1 | 1 | 1 | 3 |
| 23 | | 1 | 2 | 2 | 1 | 48 | 4 | 8 | 4 | 2 | 36 | 1 | 1 | 2 | 1 |
| 24 | 8 | 1 | 2 | 2 | 2 | 56 | 4 | 8 | 4 | 4 | 41 | 1 | 1 | 2 | 2 |
| 25 | | 2 | 1 | 1 | 1 | 40 | 8 | 4 | 4 | 2 | 32 | 1 | 1 | 1 | 1 |
| 26 | | 2 | 1 | 1 | 2 | 48 | 8 | 4 | 4 | 4 | 36 | 1 | 1 | 1 | 3 |
| 27 | | 2 | 1 | 2 | 1 | 48 | 8 | 4 | 4 | 2 | 36 | 1 | 1 | 2 | 1 |
| 28 | | 2 | 1 | 2 | 2 | 56 | 8 | 4 | 4 | 4 | 41 | 1 | 1 | 2 | 2 |
| 29 | | 2 | 2 | 1 | 1 | 48 | 8 | 8 | 4 | 2 | 38 | 1 | 1 | 1 | 1 |
| 30 | | 2 | 2 | 1 | 2 | 56 | 8 | 8 | 4 | 4 | 43 | 1 | 1 | 1 | 2 |
| 31 | | 2 | 2 | 2 | 1 | 56 | 8 | 8 | 4 | 2 | 42 | 1 | 1 | 2 | 1 |
| 32 | | 2 | 2 | 2 | 2 | 64 | 8 | 8 | 4 | 4 | 47 | 1 | 1 | 2 | 2 |

4. NUMERICAL EXAMPLES

Let K_i be the fixed cost incurred for each shipment to facility i , $i = 0, \dots, N$, which is independent of the shipment size. We used the following examples:

demand = simple Poisson, compound Poisson,
 $N = 4, 8,$
 $K_0 = 100, L_0 = 2, h_0 = 1,$
 $\lambda_i = \lambda^{\lceil 4i/N \rceil}, K_i = K^{\lceil 4i/N \rceil}, L_i = 1, h_i = 0.5, p_i = 10,$
 $i = 1, \dots, N,$

where $\lambda^j = 1, 2$ and $K^j = 32/2^j$ for $j = 1, 2, 3, 4$. (Recall that $\lceil x \rceil$ is the smallest integer greater than or equal to x . Therefore if $N = 4$, we have $\lambda_i = \lambda^i$ and $K_i = K^i$ for $i = 1, 2, 3, 4$; and if $N = 8$, we have $\lambda_1 = \lambda_2 = \lambda^1$ and $K_1 = K_2 = K^1, \lambda_3 = \lambda_4 = \lambda^2$ and $K_3 = K_4 = K^2$, etc. In other words when $N = 8$, there are four groups of identical retailers. So think of the superscript as a group index.) For each example, either all the retailers face simple Poisson demand or all face compound Poisson demand. For the latter case, we assumed

$$\Pr(D_i = d) = 0.5^d, \quad d = 1, 2, \dots,$$

for $i = 1, \dots, N$, i.e., a geometric distribution with mean 2.

The examples are listed in Tables I and II, which also specify an echelon-stock (R, nQ) policy for each example

with $R_i = R^{\lceil 4i/N \rceil}$ and $Q_i = Q^{\lceil 4i/N \rceil}$ for $i = 1, \dots, N$. These policies were obtained through a heuristic algorithm: 1) Assuming that the demand at each retailer arrives continuously at a constant rate— λ_i for simple Poisson and $2\lambda_i$ for compound Poisson—use Roundy’s (1985) algorithm to compute power-of-two order quantities; 2) Given these order quantities, search for reorder points. The second step is facilitated by Proposition 3.

The average costs of the above examples are reported in Tables III and IV. Table III contains examples with simple Poisson demand and has the following columns: exact average holding and backorder cost, approximate average holding and backorder cost, the relative error of the approximation (approximate cost/exact cost - 1), lower bound on the average costs of all feasible policies, simulated total cost (including setup cost), and the relative difference between the lower bound and the total cost (1 - lower bound/total cost). The simulated total cost was obtained by adding the exact holding and backorder cost to the simulated setup cost (since we do not have an exact formula for the average setup cost). Table IV contains examples with compound Poisson demand and has the following columns: simulated holding and backorder cost, approximate holding and backorder costs under both approximations, the relative

Table II
Examples with Compound Poisson Demand

| No. | N | λ^1 | λ^2 | λ^3 | λ^4 | Q_0 | Q^1 | Q^2 | Q^3 | Q^4 | R_0 | R^1 | R^2 | R^3 | R^4 |
|-----|---|-------------|-------------|-------------|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 33 | | 1 | 1 | 1 | 1 | 32 | 8 | 8 | 4 | 4 | 29 | 3 | 3 | 5 | 5 |
| 34 | | 1 | 1 | 1 | 2 | 40 | 8 | 8 | 4 | 4 | 33 | 2 | 2 | 4 | 9 |
| 35 | | 1 | 1 | 2 | 1 | 40 | 8 | 8 | 8 | 4 | 34 | 3 | 3 | 8 | 4 |
| 36 | | 1 | 1 | 2 | 2 | 48 | 8 | 8 | 8 | 4 | 39 | 2 | 2 | 7 | 9 |
| 37 | | 1 | 2 | 1 | 1 | 40 | 8 | 8 | 4 | 4 | 33 | 2 | 7 | 4 | 4 |
| 38 | | 1 | 2 | 1 | 2 | 48 | 8 | 8 | 4 | 4 | 38 | 2 | 7 | 4 | 9 |
| 39 | | 1 | 2 | 2 | 1 | 48 | 8 | 8 | 8 | 4 | 39 | 2 | 7 | 7 | 4 |
| 40 | 4 | 1 | 2 | 2 | 2 | 56 | 8 | 8 | 8 | 4 | 44 | 2 | 7 | 7 | 9 |
| 41 | | 2 | 1 | 1 | 1 | 40 | 16 | 8 | 4 | 4 | 35 | 5 | 3 | 5 | 5 |
| 42 | | 2 | 1 | 1 | 2 | 48 | 16 | 8 | 4 | 4 | 40 | 4 | 3 | 4 | 9 |
| 43 | | 2 | 1 | 2 | 1 | 48 | 16 | 8 | 8 | 4 | 40 | 4 | 3 | 8 | 4 |
| 44 | | 2 | 1 | 2 | 2 | 56 | 16 | 8 | 8 | 4 | 46 | 4 | 3 | 7 | 9 |
| 45 | | 2 | 2 | 1 | 1 | 48 | 16 | 8 | 4 | 4 | 40 | 4 | 7 | 4 | 4 |
| 46 | | 2 | 2 | 1 | 2 | 56 | 16 | 8 | 4 | 4 | 45 | 4 | 7 | 4 | 9 |
| 47 | | 2 | 2 | 2 | 1 | 56 | 16 | 8 | 8 | 4 | 45 | 4 | 7 | 7 | 4 |
| 48 | | 2 | 2 | 2 | 2 | 64 | 16 | 8 | 8 | 4 | 51 | 4 | 7 | 7 | 9 |
| 49 | | 1 | 1 | 1 | 1 | 64 | 8 | 8 | 4 | 4 | 57 | 3 | 3 | 4 | 4 |
| 50 | | 1 | 1 | 1 | 2 | 80 | 8 | 8 | 4 | 4 | 66 | 2 | 2 | 4 | 8 |
| 51 | | 1 | 1 | 2 | 1 | 80 | 8 | 8 | 8 | 4 | 67 | 2 | 2 | 7 | 4 |
| 52 | | 1 | 1 | 2 | 2 | 96 | 8 | 8 | 8 | 4 | 77 | 2 | 2 | 7 | 8 |
| 53 | | 1 | 2 | 1 | 1 | 80 | 8 | 8 | 4 | 4 | 66 | 2 | 7 | 4 | 4 |
| 54 | | 1 | 2 | 1 | 2 | 96 | 8 | 8 | 4 | 4 | 76 | 2 | 7 | 4 | 8 |
| 55 | | 1 | 2 | 2 | 1 | 96 | 8 | 8 | 8 | 4 | 77 | 2 | 7 | 7 | 4 |
| 56 | 8 | 1 | 2 | 2 | 2 | 112 | 8 | 8 | 8 | 4 | 86 | 3 | 7 | 7 | 9 |
| 57 | | 2 | 1 | 1 | 1 | 80 | 16 | 8 | 4 | 4 | 69 | 4 | 3 | 4 | 4 |
| 58 | | 2 | 1 | 1 | 2 | 96 | 16 | 8 | 4 | 4 | 80 | 4 | 3 | 4 | 9 |
| 59 | | 2 | 1 | 2 | 1 | 96 | 16 | 8 | 8 | 4 | 81 | 4 | 2 | 7 | 4 |
| 60 | | 2 | 1 | 2 | 2 | 112 | 16 | 8 | 8 | 4 | 91 | 4 | 2 | 7 | 9 |
| 61 | | 2 | 2 | 1 | 1 | 96 | 16 | 8 | 4 | 4 | 79 | 4 | 7 | 4 | 4 |
| 62 | | 2 | 2 | 1 | 2 | 112 | 16 | 8 | 4 | 4 | 87 | 4 | 7 | 4 | 9 |
| 63 | | 2 | 2 | 2 | 1 | 112 | 16 | 8 | 8 | 4 | 91 | 4 | 7 | 7 | 4 |
| 64 | | 2 | 2 | 2 | 2 | 64 | 8 | 8 | 8 | 4 | 114 | 6 | 6 | 6 | 8 |

errors of the approximations (approximate cost/simulated cost - 1), lower bound, simulated total cost, and the relative difference between the lower bound and the total cost (1 - lower bound/total cost). All simulated entries include a 95% confidence interval. The lower bound is the induced-penalty bound provided by Chen and Zheng (1994b).

5. CONCLUSION

This paper has provided exact as well as approximate procedures for evaluating the performance of echelon-stock (R, nQ) policies in one-warehouse multi-retailer systems. We have also presented numerical evidence on the gap between echelon-stock (R, nQ) policies and the (unknown) optimal among all feasible policies.

We have concentrated on performance evaluation of echelon-stock (R, nQ) policies. How to determine a good policy within the class remains an open question. For our numerical examples, we used a heuristic algorithm to determine reorder points and order quantities. Although this method has been suggested by many researchers, there is no guarantee that the resulting policy is close to optimal (within the class).

Finally, we note that echelon-stock policies are a very specific way of using centralized stock information. There-

fore, it is reasonable to ask how far they can be from the optimal (among all policies). Although our numerical evidence is encouraging, there is a great desire to establish a theoretical worst-case bound on the performance of the heuristic policies. So far, only preliminary results exist for two-stage serial systems (Atkins and De 1992 and Chen 1994).

APPENDIX I

Proof of Proposition 1. We first show that Z_0, \dots, Z_{N-1} are independent. Take any time epoch t . Suppose

$$\Pr(Z_i(t) = z_i, i = 0, \dots, N - 1) = 1 / \prod_{i=0}^{N-1} Q_i,$$

$$1 \leq z_i \leq Q_i, \quad i = 0, \dots, N - 1.$$

We need only to verify that the above distribution is stationary for the Markov chain $\{Z_i(t)\}_{i=0}^{N-1}$. Take any time epoch $t' > t$. Let D_i be the total demand at retailer i , and D_0 the total system demand, in the interval (t, t') . Note that

$$Z_i(t) = Z_i(t') + D_i - mQ_i \stackrel{\text{def}}{=} f_i(Z_i(t'), D_i),$$

where m is an integer so that $1 \leq Z_i(t') + D_i - mQ_i \leq Q_i$. (Thus m is unique.) Let $\mathbf{D} = (D_i)_{i=0}^{N-1}$ and $\mathbf{d} = (d_i)_{i=0}^{N-1}$. Take any z'_i with $1 \leq z'_i \leq Q_i, i = 0, \dots, N - 1$. Note that given $\mathbf{D} = \mathbf{d}, Z_i(t') = z'_i$ if and only if $Z_i(t) = f_i(z'_i, d_i)$. Therefore,

Table III

Exact Cost, Approximate Cost, Lower Bound and Simulated Total Cost for Examples with Simple Poisson Demand

| No. | Exact Cost | Approximation | | Lower Bound | Total Cost | Gap |
|-----|------------|---------------|-------|-------------|---------------|------|
| | | Cost | Error | | | |
| 1 | 31.67 | 32.67 | 0.03 | 44.47 | 50.73 ± 0.01 | 0.12 |
| 2 | 37.50 | 38.56 | 0.03 | 49.70 | 56.20 ± 0.01 | 0.12 |
| 3 | 36.80 | 37.69 | 0.02 | 49.85 | 56.80 ± 0.01 | 0.12 |
| 4 | 42.54 | 43.49 | 0.02 | 54.81 | 62.21 ± 0.01 | 0.12 |
| 5 | 37.69 | 38.96 | 0.03 | 50.13 | 56.84 ± 0.01 | 0.12 |
| 6 | 43.69 | 44.93 | 0.03 | 55.07 | 62.53 ± 0.01 | 0.12 |
| 7 | 42.91 | 44.15 | 0.03 | 55.22 | 62.99 ± 0.01 | 0.12 |
| 8 | 36.59 | 37.59 | 0.03 | 59.92 | 70.86 ± 0.01 | 0.15 |
| 9 | 35.98 | 36.85 | 0.02 | 50.49 | 57.22 ± 0.01 | 0.12 |
| 10 | 41.93 | 42.83 | 0.02 | 55.42 | 62.76 ± 0.01 | 0.12 |
| 11 | 41.11 | 41.95 | 0.02 | 55.57 | 63.27 ± 0.01 | 0.12 |
| 12 | 36.59 | 37.59 | 0.03 | 60.28 | 71.13 ± 0.01 | 0.15 |
| 13 | 42.14 | 43.35 | 0.03 | 55.84 | 63.43 ± 0.01 | 0.12 |
| 14 | 37.79 | 39.29 | 0.04 | 60.53 | 71.62 ± 0.02 | 0.15 |
| 15 | 47.34 | 48.52 | 0.02 | 60.69 | 69.55 ± 0.01 | 0.13 |
| 16 | 41.25 | 42.57 | 0.03 | 65.22 | 75.97 ± 0.01 | 0.14 |
| 17 | 48.72 | 49.90 | 0.02 | 72.92 | 90.58 ± 0.02 | 0.19 |
| 18 | 58.04 | 59.37 | 0.02 | 81.44 | 99.03 ± 0.02 | 0.18 |
| 19 | 54.75 | 55.70 | 0.02 | 81.76 | 98.55 ± 0.02 | 0.17 |
| 20 | 63.26 | 64.34 | 0.02 | 89.92 | 106.26 ± 0.02 | 0.15 |
| 21 | 57.74 | 59.66 | 0.03 | 82.31 | 99.83 ± 0.02 | 0.18 |
| 22 | 67.01 | 69.06 | 0.03 | 90.44 | 108.32 ± 0.02 | 0.17 |
| 23 | 64.48 | 66.30 | 0.03 | 90.76 | 108.47 ± 0.03 | 0.16 |
| 24 | 73.09 | 74.95 | 0.03 | 98.60 | 116.37 ± 0.02 | 0.15 |
| 25 | 57.74 | 59.66 | 0.03 | 82.98 | 100.22 ± 0.02 | 0.17 |
| 26 | 67.01 | 69.06 | 0.03 | 91.12 | 108.64 ± 0.02 | 0.16 |
| 27 | 64.48 | 66.30 | 0.03 | 91.44 | 108.77 ± 0.02 | 0.16 |
| 28 | 73.09 | 74.95 | 0.03 | 99.27 | 116.66 ± 0.02 | 0.15 |
| 29 | 66.96 | 69.41 | 0.04 | 91.99 | 109.51 ± 0.02 | 0.16 |
| 30 | 75.63 | 78.12 | 0.03 | 99.79 | 117.47 ± 0.02 | 0.15 |
| 31 | 73.58 | 75.90 | 0.03 | 100.11 | 117.98 ± 0.02 | 0.15 |
| 32 | 82.13 | 84.49 | 0.03 | 107.70 | 125.84 ± 0.02 | 0.14 |

$$\begin{aligned} & \Pr(Z_i(t') = z'_i, i = 0, \dots, N - 1 | \mathbf{D} = \mathbf{d}) \\ &= \Pr(Z_i(t) = f_i(z'_i, d_i), i = 0, \dots, N - 1 | \mathbf{D} = \mathbf{d}) \\ &= 1 / \prod_{i=0}^{N-1} Q_i. \end{aligned}$$

Unconditioning the above conditional probability, we have

$$\Pr(Z_i(t') = z'_i, i = 0, \dots, N - 1) = 1 / \prod_{i=0}^{N-1} Q_i.$$

Now suppose t is a time epoch at steady state. Let $t' = t + L_0$. Note that

$$\begin{aligned} & \Pr(IL_0(t') = w, Z_i(t') = z'_i, i = 1, \dots, N - 1 | \mathbf{D} = \mathbf{d}) \\ &= \Pr(NIP_0(t) - d_0 = w, Z_i(t) = f_i(z'_i, d_i), \\ & \quad i = 1, \dots, N - 1 | \mathbf{D} = \mathbf{d}) \\ &= \Pr(NIP_0(t) - d_0 = w, Z_i(t) = f_i(z'_i, d_i), \\ & \quad i = 1, \dots, N - 1) \\ &= \Pr(NIP_0(t) - d_0 = w) / \prod_{i=1}^{N-1} Q_i \end{aligned}$$

where the last equality is due to the fact that $Z_0(t)$ (thus $NIP_0(t)$), \dots , $Z_{N-1}(t)$ are independent. The proposition follows by unconditioning the above conditional probability. \square

APPENDIX II. COMPUTATIONAL ISSUES

II.1. A Recursive Procedure for $P_{-i}^y(n|j)$

First use (11) recursively to obtain $P_{-i}^y(n|j)$. Let $b(\lambda_i, \lambda_0; l, j) = b_i(l, j)$ and $p_i(z, y) = n(\mathcal{N} \setminus \{i\}, y - z) / n(\mathcal{N}, y)$. Recall that $p_i^z(0|0) = 1$ for $z = 1, \dots, Q_i$. From (11), we have

$$\begin{aligned} & p_i(1, y) b_i(0, j) P_{-i}^{y-1}(n|j) = P_{-i}^y(n|j) \\ & - \sum_{z=1}^{Q_i} p_i(z, y) \sum_{l=0}^j b_i(l, j) \sum_{m=1}^n p_i^z(m|l) P_{-i}^{y-z}(n - m|j - l) \\ & - \sum_{z=1}^{Q_i} p_i(z, y) \sum_{l=1}^j b_i(l, j) p_i^z(0|l) P_{-i}^{y-z}(n|j - l) \\ & - \sum_{z=2}^{Q_i} p_i(z, y) b_i(0, j) P_{-i}^{y-z}(n|j). \end{aligned} \tag{25}$$

(When a summation's lower limit exceeds its upper limit, it is void.) We can use (25) recursively to determine $P_{-i}^y(n|j)$. A similar recursive procedure can be developed for the approximation in the simple Poisson case as well as the two approximations in the compound Poisson case.

(One potential problem in using (25) is that we may have to divide the right side of (25) by a very small number in order to obtain $P_{-i}^{y-1}(n|j)$. This occurs when j is large

Table IV
 Simulated Cost, Approximate Costs, Lower Bound and Total Cost for Examples with Compound Poisson Demand

| No. | Simulated Cost | Approximation I | | Approximation II | | Lower Bound | Total Cost | Gap |
|-----|----------------|-----------------|-------|------------------|-------|-------------|---------------|------|
| | | Cost | Error | Cost | Error | | | |
| 33 | 56.55 ± 0.14 | 55.87 | -0.01 | 53.86 | -0.05 | 74.97 | 91.11 ± 0.17 | 0.18 |
| 34 | 63.35 ± 0.18 | 62.88 | -0.01 | 61.27 | -0.03 | 83.88 | 98.69 ± 0.22 | 0.15 |
| 35 | 64.31 ± 0.17 | 63.73 | -0.01 | 61.92 | -0.04 | 84.09 | 99.06 ± 0.21 | 0.15 |
| 36 | 71.11 ± 0.16 | 70.72 | -0.01 | 69.14 | -0.03 | 92.56 | 106.62 ± 0.18 | 0.13 |
| 37 | 62.94 ± 0.15 | 62.40 | -0.01 | 60.59 | -0.04 | 84.40 | 99.54 ± 0.19 | 0.15 |
| 38 | 70.61 ± 0.20 | 70.06 | -0.01 | 67.84 | -0.04 | 92.87 | 108.07 ± 0.23 | 0.14 |
| 39 | 70.80 ± 0.14 | 70.28 | -0.01 | 68.53 | -0.03 | 93.08 | 107.54 ± 0.17 | 0.13 |
| 40 | 78.52 ± 0.18 | 77.92 | -0.01 | 75.74 | -0.04 | 101.23 | 116.07 ± 0.21 | 0.13 |
| 41 | 65.34 ± 0.15 | 64.89 | -0.01 | 63.54 | -0.03 | 84.90 | 100.52 ± 0.17 | 0.16 |
| 42 | 72.68 ± 0.22 | 72.03 | -0.01 | 71.17 | -0.02 | 93.37 | 108.61 ± 0.24 | 0.14 |
| 43 | 73.32 ± 0.17 | 72.78 | -0.01 | 71.89 | -0.02 | 93.58 | 108.58 ± 0.19 | 0.14 |
| 44 | 80.55 ± 0.19 | 80.04 | -0.01 | 79.19 | -0.02 | 101.72 | 116.57 ± 0.22 | 0.13 |
| 45 | 71.92 ± 0.17 | 71.43 | -0.01 | 70.58 | -0.02 | 93.88 | 109.07 ± 0.20 | 0.14 |
| 46 | 79.69 ± 0.17 | 79.12 | -0.01 | 77.90 | -0.02 | 102.03 | 117.59 ± 0.20 | 0.13 |
| 47 | 80.20 ± 0.23 | 79.66 | -0.01 | 78.69 | -0.02 | 102.24 | 117.41 ± 0.26 | 0.13 |
| 48 | 87.49 ± 0.21 | 87.11 | -0.00 | 85.85 | -0.02 | 110.12 | 125.46 ± 0.25 | 0.12 |
| 49 | 110.39 ± 0.23 | 109.50 | -0.01 | 104.19 | -0.06 | 126.38 | 153.74 ± 0.26 | 0.18 |
| 50 | 123.75 ± 0.29 | 123.28 | -0.00 | 118.53 | -0.04 | 141.46 | 168.77 ± 0.34 | 0.16 |
| 51 | 125.42 ± 0.29 | 124.73 | -0.01 | 120.07 | -0.04 | 141.85 | 169.02 ± 0.31 | 0.16 |
| 52 | 139.72 ± 0.33 | 139.42 | -0.00 | 134.22 | -0.04 | 156.36 | 185.03 ± 0.37 | 0.15 |
| 53 | 124.08 ± 0.28 | 123.13 | -0.01 | 117.57 | -0.05 | 142.43 | 171.48 ± 0.32 | 0.17 |
| 54 | 138.32 ± 0.32 | 137.82 | -0.00 | 131.77 | -0.05 | 156.93 | 187.40 ± 0.36 | 0.16 |
| 55 | 139.82 ± 0.36 | 139.25 | -0.00 | 133.32 | -0.05 | 157.32 | 187.47 ± 0.41 | 0.16 |
| 56 | 156.15 ± 0.34 | 155.59 | -0.00 | 147.32 | -0.06 | 171.39 | 205.50 ± 0.39 | 0.17 |
| 57 | 128.40 ± 0.29 | 127.83 | -0.00 | 123.96 | -0.03 | 143.42 | 172.40 ± 0.31 | 0.17 |
| 58 | 143.58 ± 0.35 | 142.90 | -0.00 | 138.29 | -0.04 | 157.91 | 189.23 ± 0.38 | 0.17 |
| 59 | 143.73 ± 0.31 | 143.33 | -0.00 | 139.80 | -0.03 | 158.30 | 187.85 ± 0.34 | 0.16 |
| 60 | 159.05 ± 0.35 | 158.64 | -0.00 | 153.80 | -0.03 | 172.37 | 204.86 ± 0.40 | 0.16 |
| 61 | 142.66 ± 0.31 | 141.90 | -0.01 | 137.35 | -0.04 | 158.88 | 190.62 ± 0.35 | 0.17 |
| 62 | 158.52 ± 0.39 | 157.98 | -0.00 | 151.37 | -0.05 | 172.93 | 208.12 ± 0.41 | 0.17 |
| 63 | 158.77 ± 0.36 | 157.86 | -0.01 | 152.87 | -0.04 | 173.32 | 206.91 ± 0.38 | 0.16 |
| 64 | 146.51 ± 0.40 | 146.00 | -0.00 | 140.88 | -0.04 | 187.04 | 229.74 ± 0.45 | 0.19 |

and thus $b_i(0, j)$ is small. As a result, errors may be magnified. To avoid this potential problem, one can always use (11) recursively to compute $P_{-i}^y(n|j)$. However, this will increase the computational complexity of the algorithm. In solving the numerical examples in Section 4, we encountered some instances with magnified errors. However, for those instances we were able to control the errors either by increasing the precision of computer representation of real numbers or by restricting the value of j .)

II.2. Truncations

From Proposition 2, we have

$$\Pr(B_0 = b) = \sum_y n(\mathcal{N}, y) g\left(-bQ_N + \sum_{i=1}^N R_i + y\right) / \prod_{i=1}^{N-1} Q_i, \quad b = 1, 2, \dots,$$

where the summation over y is from N to $\sum_{i=1}^N Q_i$. From the above density function of B_0 , one can find a positive integer B such that $\Pr(B_0 > B) < \delta$. When δ is sufficiently small, we can use B as an upper bound on B_0 . Various truncations can be derived from this upper bound.

By (12) and (13), we need only to compute $P_{-i}^y(n|j)$ for $n \leq B$ and $P_i^{b,zy}(b_i)$ for $b_i \leq b \leq B$. Now consider $P_S^y(n|j)$. Notice that the total quantity ordered by the group S by its j th demand is greater than or equal to $j + y - \sum_{k \in S} Q_k$ and is less than or equal to $j + y - |S|$ where $|S|$ represents the number of retailers in S . Therefore, we need only to consider y, n and j that satisfy

$$j + y - \sum_{k \in S} Q_k \leq nQ_N \leq j + y - |S|.$$

Define

$$\bar{J} = BQ_N + \sum_{k=1}^{N-1} Q_k - N + 1.$$

Note that for any $n \leq B$, we have $P_{-i}^y(n|j) = 0$ if $j > \bar{J}$. As a result, we only need to compute $P_{-i}^y(n|j)$ for $j \leq \bar{J}$. Also, the summation over j in (12) has a finite range from 0 to \bar{J} .

Now consider $p_k^y(n|j)$. From (9), we know that if $j \geq nQ_N + Q_k + 1 - y$ then $p_k^y(n|j) = 0$. Since $n \leq B$, we can ignore $p_k^y(n|j)$ if $j > \underline{J}$ where

$$\underline{J} = BQ_N + \max\{Q_k, k = 1, \dots, N\}.$$

Therefore, the upper limit for the summation over l in (11) can be reduced to $\min\{j, \underline{J}\}$.

Finally, recall that $J_i(z, b_i) = \lceil b_i Q_N / Q_i \rceil Q_i + 1 - z$. Therefore for $b_i = 1, \dots, B$, we have $1 \leq J_i(z, b_i) \leq \lceil B Q_N / Q_i \rceil Q_i$. From (12), we only need to compute $b_i^m(m, j)$ for $m = 1, \dots, \lceil B Q_N / Q_i \rceil Q_i$.

II.3. Complexities

To assess the computational complexities of various evaluation procedures, (1) we assume $Q_i = Q$ for $i = 1, \dots, N$ but still follow the procedures as if the Q 's were different, (2) we assume that D_i has a negative binomial distribution for the compound Poisson case, and (3) we are not concerned with memory requirement. The input data are N, Q, B, \underline{J} and \bar{J} .

II.3.1. Simple Poisson (Exact)

Complexity: $O((NQB)^2 \underline{J} \bar{J})$

- E1a) $O(QB^2 \underline{J} \bar{J})$ Set $k \leftarrow 1$ and $S \leftarrow \{1\}$.
Set $n(S, y) = 1$ for $y = 1, \dots, Q_1$ and $n(S, y) = 0$ otherwise.
Use (10) to determine $P_S^y(n|j)$ for $y = 1, \dots, Q_1, n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$.
- E1b) $O(QB \bar{J})$ Set $k \leftarrow k + 1$. If $k \leq N$, use (9) to determine $p_k^y(n|j)$ for $y = 1, \dots, Q_k, n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$. Otherwise, go to E2a).
- E1c) $O(kQ^2)$ Use (5) to compute $n(S \cup \{k\}, y)$ for $y = |S \cup \{k\}|, \dots, \sum_{i \in S \cup \{k\}} Q_i$.
- E1d) $O(\underline{J} \bar{J})$ Compute $b(\lambda_k, \lambda_k + \lambda_S; l, j)$ for $l = 0, \dots, \min\{j, \bar{J}\}$ and $j = 0, \dots, \bar{J}$.
- E1e) $O(k(QB)^2 \underline{J} \bar{J})$ Use (11) to compute $P_{S \cup \{k\}}^y(n|j)$ for $y = |S \cup \{k\}|, \dots, \sum_{i \in S \cup \{k\}} Q_i, n = 0, \dots, B$ and $j = 0, \dots, \bar{J}$.
- E1f) Set $S \leftarrow S \cup \{k\}$. Go to E1b).
- E2a) Set $i \leftarrow 1$.
- E2b) $O(\underline{J} \bar{J})$ Compute $b_i(l, j)$ for $l = 0, \dots, \min\{j, \bar{J}\}$ and $j = 0, \dots, \bar{J}$.
- E2c) $O(NQ)$ Use (6) to compute $n(N \setminus \{i\}, y)$ for $y = N - 1, \dots, \sum_{k \neq i} Q_k$.
- E2d) $O(NQ^2)$ Compute $p_i(z, y)$ for $z = 1, \dots, Q_i$ and $y = N, \dots, \sum_{k=1}^N Q_k$.
- E2e) $O(N(QB)^2 \underline{J} \bar{J})$ Use (25) to compute $P_{-i}^y(n|j)$ for $y = N - 1, \dots, \sum_{k \neq i} Q_k, n = 0, \dots, B$ and $j = 0, \dots, \bar{J}$.
- E2f) $O(QB \bar{J})$ Compute $b_i^m(m, j)$ for $j = 0, \dots, \bar{J}$ and $m = 1, \dots, \lceil B Q_N / Q_i \rceil Q_i$.
- E2g) $O(N(QB)^2 \bar{J})$ Use (12) to compute $P_i^{b, z, y}(b_i)$ for $b_i = 0, \dots, b, b = 0, \dots, B, z = 1, \dots, Q_i$ and $y = N - 1, \dots, \sum_{k \neq i} Q_k$.
- E2h) $O(NQ^2 B)$ Use (7) and (8) to compute $\pi_i(b, z, y)$ for $b = 0, \dots, B, z = 1, \dots, Q_i, y = N - 1, \dots, \sum_{k \neq i} Q_k$.
- E2i) $O(N(QB)^2)$ Use (13) to compute $\Pr(\bar{Z}_i \leq w)$.
- E2j) Set $i \leftarrow i + 1$. If $i \leq N$, go to E2b). Otherwise, stop.

II.3.2. Simple Poisson (Approximation)

Complexity: $O(NB^2 \underline{J} \bar{J})$

- A1a) $O(B^2 \underline{J})$ Set $k \leftarrow 1$ and $S \leftarrow \{1\}$.
Use (15) to determine $P_S(n|j)$ for $n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$.
- A1b) $O(B \underline{J})$ Set $k \leftarrow k + 1$. If $k \leq N$, use (14) to determine $p_k(n|j)$ for $n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$. Otherwise, go to A2a).
- A1c) $O(\underline{J} \bar{J})$ Compute $b(\lambda_k, \lambda_k + \lambda_S; l, j)$ for $l = 0, \dots, \min\{j, \bar{J}\}$ and $j = 0, \dots, \bar{J}$.
- A1d) $O(B^2 \underline{J} \bar{J})$ Use (16) to compute $P_{S \cup \{k\}}(n|j)$ for $n = 0, \dots, B$ and $j = 0, \dots, \bar{J}$.
- A1e) Set $S \leftarrow S \cup \{k\}$. Go to A1b).
- A2a) $O((NQ)^2)$ Use (5) to compute $n(N, y)$ recursively.
Set $i \leftarrow 1$.
- A2b) $O(\underline{J} \bar{J})$ Compute $b_i(l, j)$ for $l = 0, \dots, \min\{j, \bar{J}\}$ and $j = 0, \dots, \bar{J}$.
- A2d) $O(B^2 \underline{J} \bar{J})$ Use a procedure similar to (25) to compute $P_{-i}(n|j)$ for $n = 0, \dots, B$ and $j = 0, \dots, \bar{J}$.
- A2e) $O(QB \bar{J})$ Compute $b_i^m(m, j)$ for $j = 0, \dots, \bar{J}$ and $m = 1, \dots, \lceil B Q_N / Q_i \rceil Q_i$.
- A2f) $O(QB^2 \bar{J})$ Use (17) to compute $P_i^{b, z}(b_i)$ for $b_i = 0, \dots, b, b = 0, \dots, B$ and $z = 1, \dots, Q_i$.
- A2g) $O(NQ)$ Use (6) to compute $n(N \setminus \{i\}, y)$.
- A2h) $O(NQ^2 B)$ Use (18) to compute $\pi_i(b, z)$ for $b = 0, \dots, B$ and $z = 1, \dots, Q_i$.
- A2i) $O(QB^2)$ Use (19) to compute $\Pr(\bar{Z}_i \leq w)$.
- A2j) Set $i \leftarrow i + 1$. If $i \leq N$, go to A2c). Otherwise, stop.

II.3.3. Compound Poisson (Approximation I)

Complexity: $O((NQB)^2 \underline{J} \bar{J})$

The algorithm is essentially the same as the one for simple Poisson (exact). The following are the minor changes where E2f+) should be added at the end of step E2f).

- E1a) $O(\underline{J}^2 + QB^2 \underline{J})$ Set $k \leftarrow 1$ and $S \leftarrow \{1\}$.
Set $n(S, y) = 1$ for $y = 1, \dots, Q_1$ and $n(S, y) = 0$ otherwise.
Compute $F_1^j(d)$ for $j = 1, \dots, \underline{J}$ and $d = j, \dots, \bar{J}$.
Use (20) to compute $p_1^y(n|[j])$ for $y = 1, \dots, Q_1, n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$.
Use (21) to compute $P_{\{1\}}^y(n|[j])$ for $y = 1, \dots, Q_1, n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$.
- E1b) $O(\underline{J}^2 + QB \underline{J})$ Set $k \leftarrow k + 1$.
Compute $F_k^j(d)$ for $j = 1, \dots, \underline{J}$ and $d = j, \dots, \bar{J}$.
Use (20) to compute $p_k^y(n|[j])$ for $y = 1, \dots, Q_k, n = 0, \dots, B$ and $j = 0, \dots, \underline{J}$.

- E1e) Change (11) to (22) and $PP_{S \cup \{k\}}^v(n|j)$ to $P_{S \cup \{k\}}^v(n|j)$.
- E2e) Change $P_{-i}^v(n|j)$ to $P_{-i}^v(n|j)$.
- E2f+) $O((BQ)^2)$ Use (23) to compute $\alpha_i(m, M)$ for $m = 1, \dots, M$ and $M = 1, \dots, \lceil BQ_N/Q_i \rceil Q_i$.
- E2g) $O(N(QB)^2J\bar{J})$ Change (12) to (24).

II.3.4. Compound Poisson (Approximation II)

Complexity: Same as simple Poisson (approximation)

APPENDIX III

Given $B_0(t) = b > 0$, $Z_i(t) = z$ and $Y_{-i}(t) = y$, we have from (1) that $IL_0(t) = -bQ_N + \sum_{i=1}^N R_i + z + y$. Here we demonstrate that the value of $IL_0(t)$ contains information about the demand process before time t that is relevant to the disaggregation of warehouse backorders. To this end, consider a special case where there are only two retailers and every facility uses a base-stock policy. Let S_0 be the system base-stock level. Since $S_0 - D_0(t - L_0, t) = IL_0(t)$, the value of $IL_0(t)$ determines the system demand in the interval $(t - L_0, t)$. Now suppose $D_0(t - L_0, t) = d$. Let $f_i(n, x)$ be the probability that n customers arrive at retailer i in the interval $(t - L_0, t)$ and demand a total of x units. Consider an arbitrary customer in the interval $(t - L_0, t)$. Note that

$$\begin{aligned} & \Pr(\text{the customer belongs to retailer 1} | d) \\ &= \sum_{m,n,x,y:x+y=d} \frac{m}{m+n} f_1(m, x) f_2(n, y) / \\ & \quad \sum_{i,j,u,v:u+v=d} f_1(i, u) f_2(j, v). \end{aligned}$$

In general, the above probability is different from $\lambda_1/(\lambda_1 + \lambda_2)$. (The same is true for retailer 2.) To see this, consider the simplest case with $d = 1$. Let p_{ij} be the probability that a customer at retailer i demands j units, $i = 1, 2, j = 1, 2, \dots$. Since $d = 1$, there is exactly one customer in the interval $(t - L_0, t)$. Furthermore,

$$\begin{aligned} & \Pr(\text{the customer belongs to retailer 1} | d = 1) \\ &= \lambda_1 p_{11} / (\lambda_1 p_{11} + \lambda_2 p_{21}). \end{aligned}$$

Moreover, the demand size of the customer is exactly one unit. (One implication of the above is that the disaggregation method suggested by Graves for base-stock policies is not exact for compound Poisson demand.)

On the other hand, if demand is simple Poisson then the value of $IL_0(t)$ does not provide any information about the demand process before time t that is relevant to the disaggregation of warehouse backorders. This is because for simple Poisson demand, we have

$$\begin{aligned} & \Pr(\text{a customer in } (t - L_0, t) \text{ belongs to retailer 1} | d) \\ &= \sum_{m+n=d} \frac{m}{d} \frac{\lambda_1^m}{m!} \frac{\lambda_2^n}{n!} / \left(\sum_{i+j=d} \frac{\lambda_1^i}{i!} \frac{\lambda_2^j}{j!} \right) = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

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