

OPTIMAL INVESTMENT POLICIES FOR A FIRM WITH A RANDOM RISK PROCESS: EXPONENTIAL UTILITY AND MINIMIZING THE PROBABILITY OF RUIN

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We consider a firm that is faced with an uncontrollable stochastic cash flow, or *random risk* process. There is one investment opportunity, a risky stock, and we study the optimal investment decision for such firms. There is a fundamental *incompleteness* in the market, in that the risk to the investor of going bankrupt cannot be eliminated under any investment strategy, since the random risk process ensures that there is always a positive probability of ruin (bankruptcy). We therefore focus on obtaining investment strategies which are optimal in the sense of minimizing the risk of ruin. In particular, we solve for the strategy that maximizes the probability of achieving a given upper wealth level before hitting a given lower level. This policy also minimizes the probability of ruin. We prove that when there is no risk-free interest rate, this policy is equivalent to the policy that maximizes utility from terminal wealth, for a fixed terminal time, when the firm has an exponential utility function. This validates a longstanding conjecture about the relation between minimizing probability of ruin and exponential utility. When there is a positive risk-free interest rate, the conjecture is shown to be false. We also solve for the optimal policy for the related objective of minimizing the expected discounted penalty paid upon ruin.

1. Introduction. In this paper we study the following problem: a firm, such as a property-liability insurance company, or a pension-fund management company, is faced with a risk that cannot be traded away in the marketplace, i.e., it has undertaken to meet the obligations that accrue as a stochastic process, which we will denote by $\{Y_t; t \geq 0\}$. This risk process might go negative, as for example in property-liability insurance, where Y_t denotes the net of (cumulative) premiums minus claims up to time t . For simplicity, and without any loss of generality, we will assume that there is only one investment opportunity available to this firm, a risky stock. We will not consider the uninteresting case where the risk process is perfectly correlated with the risky stock. Therefore we are in a fundamentally “incomplete market” since the risk process is in fact an uninsurable risk. Our goal is to solve for optimum investment strategies for such firms, for objectives described below. We will work in the continuous-time setting.

For the case of an *ordinary* investor in continuous-time, by which we mean an investor *without* an external risk process, i.e., $Y_t = 0$, for all t , investment strategies that maximize utility of terminal wealth for a variety of utility functions are studied in Merton (1971, 1990), Karatzas (1989) and many others. While utility functions are subjective, there are of course specific utility functions that have many objective criteria associated with them. Most prominently, the policy that maximizes *logarithmic utility* of terminal wealth has been shown to be identical to the policy that is optimal for the following objectives: (i) maximizing the ‘growth rate’ of wealth; (ii) minimizing

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the time to achieve a given goal as well as (iii) being optimal in a competitive game-theoretic sense. (For (i), (ii), see for example Kelly (1956), Breinan (1961), Thorp (1969) for the discrete-time setting, and Pestien and Sudderth (1985), Karatzas (1989), Merton (1990) for continuous-time. See Bell and Cover (1980) for (iii) in discrete-time.) Maximizing logarithmic utility for an ordinary investor leads to (constant) proportional, or fixed fractional investment policies, whereby the investor always invests a fixed fraction of his wealth in the risky stock. This is also commonly called the *Kelly criterion* in honor of his seminal paper (Kelly (1956)). In continuous-time, when the price of the risky stock follows a diffusion, it is clear that the probability of ruin is zero under such policies.

Unfortunately, these results only hold for the case of an ordinary investor who does not have a stochastic cash flow, and not for the model under consideration here. In fact, for our model, since the risk process, $\{Y_t\}$, can go negative, *ruin* is a very real possibility, and so we take as our prime objective the avoidance of ruin.

Ferguson (1965) studied investment policies that minimize ruin for a discrete-time and discrete-space *ordinary* investor, and conjectured (Ferguson (1965) §4) that just as maximizing logarithmic utility is intrinsically related to the objective of maximizing growth, maximizing *exponential* utility from terminal wealth is intrinsically related to the objective criteria of maximizing *survival*. In particular, that maximizing utility of wealth at a fixed terminal time for an exponential utility function is asymptotically optimal for the intrinsic criteria of maximizing the *probability* of survival, which is equivalent to minimizing the probability of ruin. This was conjectured under the assumption that the investor was allowed to borrow an unlimited amount of money, and there was no risk-free interest rate.

While we are not concerned about the validity of the conjecture for the case of an ordinary investor, since as noted above, ruin can be avoided with probability one in continuous-time in that case, it turns out that the conjecture of Ferguson (1965) is true in a very strong sense for investors facing a random risk process, as we show in the sequel. To that end, we first study a model without an interest rate, and where we allow the firm to borrow an unlimited amount. In §3 we first find the strategy that maximizes utility from terminal wealth, for a fixed terminal time, when the firm has an exponential utility function. (This is of independent interest since exponential utility is actually used in determining ‘fair’ premiums by many property-liability insurance companies, see Goovaerts et al. (1990, III.6).) In §4, we then solve for the policy that *minimizes the probability of ruin* directly, and the two are seen to be equivalent, validating the conjecture, at least for the specific model considered here. The resulting policy is also optimal for the objective of *maximizing the probability of reaching any given wealth level prior to hitting a given lower wealth level*. Our result is quite surprising in that this optimal policy invests a fixed constant amount, regardless of the level of wealth the company has. This indicates that certain strategies, such as the constant *proportional* schemes typically employed in investment policies, may be inappropriate for the case where there is stochastic cash flow, or risk process. (This may help explain some of the causes behind some bankruptcies in insurance companies as well as pension funds, who typically use these proportional strategies, which our results indicate might be too “ruin prone.”) We obtain these results by solving the classical Hamilton-Jacobi-Bellman (HJB) equations of stochastic control (see, e.g., Fleming and Rishel (1975) or Krylov (1980)), which in our cases, have explicit solutions. In §5 we consider the problem of minimizing the probability of ruin under the constraint that the firm may not borrow any money to invest. Using the “smooth pasting” conditions on the HJB equations, which we are again able to solve explicitly, we find that the optimal policy in this case has two regions, where in one region we invest all our wealth in the risky stock, while in the other region, we only invest a

fixed constant. Thus our results are intermediate to the cases of ‘bold’ and ‘timid’ play. In §6, we study an unconstrained problem with a discount rate, and where there is a penalty to be paid upon ruin. We find the optimal policy to minimize the expected discounted bankruptcy penalty paid.

Finally, in §7, we generalize the model by including an interest rate, and again consider the (unconstrained) problems of maximizing exponential utility from terminal wealth, and of minimizing the probability of ruin. For both problems we find the optimal policies explicitly, and they are both no longer constant when there is a positive interest rate. However, for the former problem the optimal policy is independent of the level of wealth, while for the latter the optimal policy does depend on the wealth level (in a rather complicated way, see Theorem 6 below), and thus we find that the conjecture does not hold for this case.

In the next section, we describe the model.

2. The model. Without loss of generality, we assume that there is only one risky stock available for investment (e.g., a mutual fund), whose price at time t will be denoted by P_t . As is quite standard (see, e.g., Merton (1971, 1990), Pliska (1986)), we will assume that the price process of the risky stock satisfies the stochastic differential equation

$$(1) \quad dP_t = P_t \cdot dZ_t$$

where Z_t is a Brownian motion with drift μ and diffusion parameter σ , i.e.,

$$(2) \quad dZ_t = \mu dt + \sigma dW_t^{(1)},$$

where μ and σ are constants and $\{W_t^{(1)}: t \geq 0\}$ is a standard Brownian motion. Thus the risky stock price follows a geometric Brownian motion.

We are concerned with investment behavior in the presence of a stochastic cash flow, or a risk process, which we will denote by $\{Y_t: t \geq 0\}$, which we will assume is another (possibly correlated) Brownian motion with drift α and diffusion parameter β , i.e., Y_t satisfies the stochastic differential equation

$$(3) \quad dY_t = \alpha dt + \beta dW_t^{(2)},$$

where α and β are constants (with $\beta > 0$), and $\{W_t^{(2)}: t \geq 0\}$ is another standard Brownian motion. We will allow the two Brownian motions to be *correlated*, and we will denote their correlation coefficient by ρ , i.e., $E(W_t^{(1)}W_t^{(2)}) = \rho t$. We will not consider the uninteresting case of $\rho^2 = 1$, in which case there would only be one source of randomness in the model.

For example, if the company under consideration is a property-casualty insurance company, then $\{Y_t\}$ represents the net claims process (premium input minus claims output). In the classical theory of risk, the “true” net claims process, say $\{\hat{Y}_t\}$, is usually modeled as (see Gerber (1979), Grandell (1991))

$$\hat{Y}_t = pt - \sum_{i=1}^{N_t} C_i,$$

where p is the premium income per unit time, N_t is the number of claims up to time t (usually modeled as a stationary renewal process with rate λ) and C_i is the size of the i th claim, with $\{C_i: i \geq 1\}$ assumed to be an i.i.d. sequence. In this case, there has been much work on diffusion approximations (Iglehart (1969), Harrison (1977),

Grandell (1991)) for the net claims process, and the interpretation (at least approximately) of the parameters in (3) is

$$\alpha = p - \lambda E(C_1), \quad \beta^2 = \lambda E(C_1^2),$$

so the parameter α is to be understood as the relative safety loading of the claims process.

The company is allowed to invest in the risky stock, and we will denote the total amount of money invested in the risky stock at time t under an investment policy f as f_t , where $\{f_t\}$ is a suitable, admissible adapted control process, i.e., f_t is a nonanticipative function that satisfies $\int_0^T f_t^2 dt < \infty$, a.s., for every $T < \infty$.

Let X_t^f denote the *wealth* of the company at time t , if it follows policy f , with $X_0 = x$. This process then evolves as

$$(4) \quad dX_t^f = f_t dZ_t + dY_t, \quad X_0 = x,$$

where Z_t and Y_t were previously defined. Substituting from (2) and (3) shows that the wealth process will follow the (controlled) stochastic differential equation

$$(5) \quad dX_t^f = (f_t \mu + \alpha) dt + f_t \sigma dW_t^{(1)} + \beta dW_t^{(2)}, \quad X_0 = x.$$

Since $W_t^{(1)}$ and $W_t^{(2)}$ are correlated standard Brownian motions, with correlation coefficient ρ , the quadratic variation (see Karatzas and Shreve (1988)) of the wealth process satisfies

$$(6) \quad d\langle X \rangle_t = (f_t^2 \sigma^2 + \beta^2 + 2\rho\sigma\beta f_t) dt,$$

and thus (for Markov control processes f) the generator is

$$(7) \quad \mathcal{A}^f g(t, x) = g_t + [f\mu + \alpha]g_x + \frac{1}{2}[f^2\sigma^2 + \beta^2 + 2\rho\sigma\beta f]g_{xx}.$$

Please note that so long as $\rho^2 \neq 1$, this model is *incomplete* in a very strong sense in that the random cash flow, or *random endowment*, Y cannot be traded on the security market, and therefore the risk to the investor cannot be eliminated under any circumstances. This incompleteness, as well as the fact that since Y is a Brownian motion, wealth can go *negative*, is what differentiates our model from those considered in Merton (1971, 1990), Karatzas (1989).

Usually, we will put no constraints on the control f_t , in particular, in §§3, 4, 6 and 7, we will allow $f_t < 0$, as well as $f_t > X_t^f$ (a similar assumption was made in Ferguson (1965, §4), for a much simpler discrete-time problem for an ordinary investor). In the first instance, the company is shorting the stock, while in the second instance it is borrowing money to invest long in the stock. (This is not an entirely unrealistic situation, in that as long as a company has a *positive* net worth, i.e., $X_t^f > 0$, it can usually borrow money.) We do not allow the company to borrow money once it is bankrupt, and thus the possibility of *ruin* is of real concern. In §5, we will consider the problem of maximizing survival for the constrained case, where we do not allow any borrowing, so that we restrict our attention to policies f such that $0 \leq f_t \leq X_t^f$.

3. Maximizing exponential utility of terminal wealth. Ferguson (1965) considered a discrete-time and space *ordinary* investor ($\alpha = \beta = 0$, i.e., no external risk process) facing a favorable investment, and found that when the investor has an *exponential utility* function, say $u(x) = -e^{-\theta x}$, and is interested in maximizing the

utility of his terminal fortune at a fixed terminal time, the optimal policy was to invest a fixed constant (Ferguson (1965, pp. 180–181)). It was conjectured there that such a strategy was *asymptotically* optimal in general for the criteria of maximizing the probability of “survival,” or minimizing the probability of ruin, for “some value of θ .” We will prove a stronger form of this conjecture in continuous-time for a more complicated model than was considered there. We will in fact show that the policy that maximizes exponential utility of terminal wealth at a fixed terminal time is *exactly* equivalent to the policy that minimizes the probability of ruin, for a specific value of θ .

To proceed, suppose now that the investor is interested in maximizing the utility from his terminal wealth, say at time T . The utility function is $u(x)$, where we assume that $u' > 0$, and $u'' < 0$. Let $V(t, x)$ denote the maximal utility attainable by the investor from the state x at time t , i.e., $V(t, x) = \sup_f E[u(X_T^f) | X_t^f = x]$, and let $\{f_t^* : 0 \leq t \leq T\}$ denote the optimal policy.

Suppose now that the investor has an exponential utility function

$$(8) \quad u(x) = \lambda - \frac{\gamma}{\theta} e^{-\theta x}$$

where $\gamma > 0$, and $\theta > 0$. This utility has (Pratt (1964)) *constant absolute risk aversion* parameter θ , as can be seen from the fact that $-u''(x)/u'(x) = \theta$. Such utility functions play a prominent role in insurance mathematics and actuarial practice, since they are the only utility functions under which the principle of “zero utility” gives a fair premium that is independent of the level of reserves of an insurance company (see Gerber (1979, p. 68), or Goovaerts et al. (1990, §II.6)). For this case we have

THEOREM 1. *The optimal policy to maximize expected utility at a terminal time T is to invest, at each time $t \leq T$, the constant amount*

$$(9) \quad f^* = \frac{\mu}{\sigma^2 \theta} - \frac{\rho \beta}{\sigma},$$

and then the optimal value function is

$$(10) \quad V(t, x) = \lambda - \frac{\gamma}{\theta} \exp\{-\theta x + (T - t) \cdot Q(\theta)\},$$

where $Q(\cdot)$ is the quadratic function defined by

$$(11) \quad Q(\theta) = \theta^2 \frac{1}{2} \beta^2 (1 - \rho^2) - \theta \left(\alpha - \frac{\rho \beta \mu}{\sigma} \right) - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2.$$

PROOF. For the problem of maximizing utility from terminal wealth at a fixed terminal time T , the HJB equations become, for $t < T$ (see Fleming and Rishel (1975, §6.3), or Fleming and Soner (1993, §III.8)),

$$(12) \quad \sup_f \{\mathcal{A}^f V(t, x)\} = 0, \quad V(T, x) = u(x),$$

where $V(t, x) = \sup_f E^{t,x}[u(X_T^f)]$. In other words, for each (t, x) , we must solve the nonlinear partial differential equation of (12), and then find the value $f_t(x)$ which

maximizes the function

$$(13) \quad V_t + [f\mu + \alpha]V_x + \frac{1}{2}[f^2\sigma^2 + \beta^2 + 2\rho\sigma\beta f]V_{xx}.$$

Let us assume that the HJB equation of (12) has a classical solution V , which satisfies $V_x > 0, V_{xx} < 0$. Then differentiating with respect to f in (13) gives the optimizer

$$(14) \quad f_t^* = -\frac{\mu}{\sigma^2} \left(\frac{V_x}{V_{xx}} \right) - \frac{\rho\beta}{\sigma}.$$

When this is placed back into (13), the HJB equation (12) becomes, after some simplification, equivalent to the following nonlinear Cauchy problem for the value function V :

$$(15) \quad V_t + \left[\alpha - \frac{\rho\beta\mu}{\sigma} \right] V_x - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 \frac{V_x^2}{V_{xx}} + \frac{1}{2} \beta^2 (1 - \rho^2) V_{xx} = 0, \quad \text{for } t < T,$$

$$(16) \quad V(T, x) = u(x).$$

Note that the partial differential equation in (15) differs quite markedly from the usual case of utility maximization for an *ordinary* investor (when $\alpha = \beta = \rho = 0$), as studied for example in Merton (1971, 1990) and Karatzas (1989) and others.

We are interested in the particular case $u(x) = \lambda - (\gamma/\theta)e^{-\theta x}$. To solve the partial differential equation in (15), try to fit a solution of the form

$$(17) \quad V(t, x) = \lambda - \frac{\gamma}{\theta} \exp\{-\theta x + g(T - t)\},$$

where $g(\cdot)$ is a suitable function, and note that for this trial solution we have

$$(18) \quad \begin{aligned} V_t(t, x) &= [V(t, x) - \lambda] \cdot [-g'(T - t)], \\ V_x(t, x) &= [V(t, x) - \lambda] \cdot [-\theta], \\ V_{xx}(t, x) &= [V(t, x) - \lambda] \cdot [\theta^2]. \end{aligned}$$

The boundary condition $V(T, x) = \lambda - (\gamma/\theta)e^{-\theta x}$, implies that we must have $g(0) = 0$, and inserting (18) into (15), and canceling like terms shows that we require that $g(T - t)$ satisfy

$$\begin{aligned} 0 &= -g'(T - t) + \frac{1}{2} \beta^2 (1 - \rho^2) \theta^2 - \left(\alpha - \frac{\rho\beta\mu}{\sigma} \right) \theta - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 \\ &\equiv -g'(T - t) + Q(\theta). \end{aligned}$$

Integrating and setting $g(0) = 0$ then gives the value function of (10).

Since we have the value function in explicit form, it is now a simple matter to obtain the optimal control. Specifically, substituting the values for V_x and V_{xx} from (18) into (14) gives the optimal control of (9).

It remains to *verify* that the control and value function above are in fact optimal. To do this, first note that under the policy given above, the resulting optimal wealth process, $\{X_t^*\}$ is a Brownian motion with constant drift and diffusion coefficients, in

particular, it is stochastically equivalent to the process that evolves as

$$(19) \quad dX_t^{f^*} = \left[\frac{\mu^2}{\sigma^2 \theta} + \left(\alpha - \frac{\rho \beta \mu}{\sigma} \right) \right] dt + \sqrt{\left(\frac{\mu}{\sigma \theta} \right)^2 + \beta^2 (1 - \rho^2)} dW_t,$$

where $\{W_t: t \geq 0\}$ is another standard Brownian motion. (This can be obtained by placing the control (9) back into the evolutionary stochastic differential equation for the wealth process and then simplifying.)

Therefore, since the value function is twice continuously differentiable, it is clear that all the conditions of the classical verification theorems (e.g., Fleming and Rishel (1975, Theorem 4.2), or Fleming and Soner (1993, Theorem III.8.1)) are satisfied, and that these are in fact the optimal controls and value function.

Alternatively, we may use the *martingale optimality principle*, by showing that under policy f^* , the value function $V(t, x)$ of (10) is a martingale. This can be established directly from (19), from which we see that, for $s \leq T - t$,

$$(20) \quad E(\exp\{-\theta(X_{t+s}^{f^*} - X_t^{f^*})\} | X_t^{f^*}) = \exp\{s \cdot Q(\theta)\},$$

from which it is sufficient to conclude that $V(t, X_t^{f^*})$ is indeed a martingale. It can then be shown that $V(t, X_t^f)$ is a supermartingale under any other admissible policy, f , which establishes optimality. \square

REMARK 1. One objection to using exponential utility in practice is that it admits a policy under which wealth can go negative. This is a valid objection for an ordinary investor, in that there are alternative utility functions under which the probability of ruin is zero almost surely (Breiman (1961), Thorp (1969), Ferguson (1965)). However, for the model under consideration here, ruin can *never* be avoided with probability one (since Y_t is a Brownian motion), so this objection is not applicable here. An ordinary investor in discrete-time with an exponential utility function, and a bankruptcy penalty was studied in Lippman et al. (1980).

4. Maximizing probability of survival, or minimizing probability of ruin. In this section, we solve directly for the policy that minimizes the probability of ruin. To that end, let τ_z^f denote the first hitting time to z under policy f , i.e., for any z ,

$$\tau_z^f = \inf\{t > 0: X_t^f = z\},$$

and for the particular two numbers a and b , let $\tau^f = \min\{\tau_a^f, \tau_b^f\}$.

Our goal is to solve for the optimal policy that maximizes $P(X_{\tau^f}^f \geq b | X_0 = x) \equiv P(\tau^f = \tau_b^f | X_0 = x)$, where $a < x < b$. For the case of an ordinary investor in continuous-time, with $\alpha = \beta = 0$, this problem (among others) was first solved in Pestien and Sudderth (1985) where f_t was restricted to lie in a given set. Analogous to the discrete-time problem studied in the classic book of Dubins and Savage (1965, 1976), Pestien and Sudderth (1985) proved that *bold* play is optimal for $\mu < 0$, and *timid* play is optimal for $\mu > 0$. For the model under consideration here, the optimal policy is intermediate to this, and was quite unexpected, in that it turns out that for the case where there *is* a stochastic cash flow, the optimal policy is to invest a *constant amount*, regardless of the size of wealth or the signs of the parameters.

Specifically, suppose now that we start off at the point $X_0 = x$, with $0 \leq a < x < b \leq \infty$, and our objective is to maximize the *probability* of reaching the point b before hitting the "ruin" point a . For reasons that will become clear soon, define the

constants η^+, D, C by

$$(21) \quad \eta^+ \equiv \frac{(\alpha - \rho\beta\mu/\sigma) + \sqrt{D}}{\beta^2(1 - \rho^2)}.$$

$$(22) \quad D \equiv \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)^2 + \beta^2(1 - \rho^2)\left(\frac{\mu}{\sigma}\right)^2,$$

$$(23) \quad C \equiv \frac{\mu}{\sigma^2\eta^+} - \frac{\rho\beta}{\sigma},$$

and note that $\eta^+ \geq 0$. We will prove the following, rather surprising, result.

THEOREM 2. *The optimal policy is to always invest the fixed constant*

$$(24) \quad f^* = C,$$

regardless of the level of wealth, or the values of the parameters a and b .

REMARK 2. The policy f^* is independent of the values of the terminal points a and b , thus by letting $b \uparrow \infty$, and $a \downarrow 0$, it is clear that this is the policy that *minimizes the probability of ruin*. Therefore this theorem obviously validates the conjecture in Ferguson (1965) for our model, since the particular choice of the risk aversion parameter, $\theta = \eta^+$ in the utility function (8) of Theorem 1, gives rise to the identical policy as that obtained in (24), as then the optimal policy of Theorem 1 is $f^* = C$, where C is defined in (23).

PROOF. While we could prove this from a more general theorem in Pestien and Sudderth (1985), which does not make use of the HJB equations, we will establish it from first principles using the technique of dynamic programming. There are two advantages to this approach, first, as a by-product of using the HJB approach, we obtain the optimal value function explicitly, and secondly, we can illustrate clearly the relation between minimizing the ruin probability and maximizing exponential utility.

Let $\tau_z^f = \inf\{t > 0: X_t^f = z\}$, and let $\tau^f = \min\{\tau_a^f, \tau_b^f\}$. Our objective is to find a control policy f to maximize $P(X_{\tau^f}^f = b | X_0 = x)$, and we will let $V(x)$ denote the optimal value function, i.e., $V(x) = \sup_f P(X_{\tau^f}^f \geq b | X_0 = x)$.

From standard arguments in stochastic control (see Krylov (1980, pp. 23–25) or Fleming and Rishel (1975, VI.2)), the Hamilton-Jacobi-Bellman optimality equation is then

$$(25) \quad 0 = \sup_f \mathcal{A}^f V(x),$$

with the boundary conditions

$$(26) \quad V(a) = 0, \quad V(b) = 1.$$

Since $V(x)$ is independent of time, (7) shows that (25) is equivalent to

$$(27) \quad \max_f \left[(f\mu + \alpha)V_x + \frac{1}{2}(\sigma^2 f^2 + \beta^2 + 2\rho\sigma\beta f)V_{xx} \right] = 0, \quad \text{for } a \leq x \leq b.$$

Assuming that the HJB equation has a classical solution V that satisfies $V_{xx} < 0$, optimizing with respect to f in (27) shows that the optimal control is given by

$$(28) \quad f^*(x) = -\frac{\mu}{\sigma^2} \frac{V_x}{V_{xx}} - \frac{\rho\beta}{\sigma},$$

which when placed back into (27) yields the nonlinear differential equation for the value function

$$(29) \quad \frac{1}{2}\beta^2(1-\rho^2)V_{xx} - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\frac{V_x^2}{V_{xx}} + \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)V_x = 0, \quad \text{for } a < x < b$$

subject to the conditions $V(a) = 0, V(b) = 1$. To make this useful therefore, we need to solve (29) for the value function, and then substitute the solution back into (28) to obtain the (candidate) optimal control.

To that end, let us try a solution (to (29)) of the form

$$(30) \quad V(x) = \kappa - \frac{\delta}{\eta}e^{-\eta x},$$

with $V_x = \delta e^{-\eta x}, V_{xx} = -\eta\delta e^{-\eta x}$.

Substituting (30) into (29), shows that for (30) to be a solution to (29), we require $Q(\eta) = 0$, where $Q(\eta)$ is the quadratic

$$(31) \quad Q(\eta) = \eta^2\frac{1}{2}\beta^2(1-\rho^2) - \eta\left(\alpha - \frac{\rho\beta\mu}{\sigma}\right) - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2,$$

and we will denote the positive and negative roots of this by η^+, η^- . (Note that the discriminant of this quadratic is $D = (\alpha - \rho\beta\mu/\sigma)^2 + \beta^2(1-\rho^2)(\mu/\sigma)^2 \geq 0$, and that

$$\eta^{+,-} = \frac{(\alpha - \rho\beta\mu/\sigma) \pm \sqrt{D}}{\beta^2(1-\rho^2)},$$

and that $(\alpha - \rho\beta\mu/\sigma) \leq \sqrt{D}$, therefore in fact, $\eta^+ > 0$, and $\eta^- < 0$ for $\rho^2 \neq 1$, regardless of the signs and magnitudes of the parameters μ, α, ρ .)

Thus we have found two solutions to (29), and it remains to determine which (if any) is applicable.

The boundary condition $V(a) = 0$ determines the constant κ as

$$(32) \quad \kappa = \frac{\delta}{\eta}e^{-\eta a},$$

and the boundary condition $V(b) = 1$ determines the constant δ as

$$(33) \quad \delta = \frac{\eta}{e^{-\eta a} - e^{-\eta b}}.$$

Since we need $V_x > 0, V_{xx} < 0$ for (28) to be (a candidate) optimal, we see that we need the *positive* root, η^+ in the above (as then, $\delta > 0, -\eta^+\delta < 0$).

Therefore, substituting (32) and (33) into (30), we find that the optimal value function is explicitly

$$(34) \quad V(x) = \frac{e^{-\eta^+ a} - e^{-\eta^+ x}}{e^{-\eta^+ a} - e^{-\eta^+ b}},$$

and when we place V_x, V_{xx} back into (28), noting that $V_x/V_{xx} = -1/\eta^+$, we find that the optimal policy is

$$f^* = \left[\frac{\mu}{\sigma^2 \eta^+} - \frac{\rho \beta}{\sigma} \right].$$

Thus the optimal policy always invests the *constant* amount f^* .

It remains to *verify* that f^* is in fact the optimal policy. To do this, first recognize the value function of (34) as the probability of hitting b before a from the initial state x for a Brownian motion with scale density $s(z) = e^{-\eta^+ z}$. In fact, under policy f^* , the wealth process is a Brownian motion with a *positive* drift, i.e.,

COROLLARY 1. *Under the control policy f^* , the wealth process, $X_t^{f^*}$ is equal in distribution to X_t^* , where X_t^* is a Brownian motion with positive drift. In particular,*

$$(36) \quad dX_t^* = \left[\frac{\mu^2}{\sigma^2 \eta^+} + \left(\alpha - \frac{\rho \beta \mu}{\sigma} \right) \right] dt + \sqrt{\left(\frac{\mu}{\sigma \eta^+} \right)^2 + \beta^2 (1 - \rho^2)} dW_t,$$

where $\{W_t; t \geq 0\}$ is another standard Brownian motion.

PROOF. Simply place the (optimal) policy f^* back into the stochastic differential equation for the wealth process (5), to obtain $dX_t^{f^*} = (f^* \mu + \alpha) dt + f^* \sigma dW_t^{(1)} + \beta dW_t^{(2)}$, and note that the quadratic variation of this semi-martingale satisfies

$$d\langle X^{f^*} \rangle_t = (f^{*2} \sigma^2 + \beta^2 + 2\rho \sigma \beta f^*) dt.$$

Substituting for f^* from (34) then yields the diffusion displayed in (36). The distributional equality follows since the generators of the two processes are the same.

To check that the drift of this Brownian motion is positive, regardless of the signs of α, μ, ρ , let

$$\omega = \alpha - \frac{\rho \beta \mu}{\sigma}, \quad \nu = \beta^2 (1 - \rho^2) \left(\frac{\mu}{\sigma} \right)^2,$$

and note that $\nu > 0$ for $\rho^2 \neq 1$. We may therefore write the drift of the Brownian motion as

$$(37) \quad \frac{\mu^2}{\sigma^2 \eta^+} + \left(\alpha - \frac{\rho \beta \mu}{\sigma} \right) = \frac{\sqrt{\omega^2 + \nu}}{\eta^+} (\sqrt{\omega^2 + \nu} + \omega),$$

which is clearly positive, regardless of the sign of ω , since $\sqrt{\omega^2 + \nu} + \omega > 0$ for $\nu > 0$. \square

To return to the verification of optimality, recognize first that under f^* , X^* has constant coefficients, and that $P_x(\tau^{f^*} < \infty) = 1$. Furthermore, the value function is twice continuously differentiable, with V_{xx} obviously satisfying a Lipschitz condition. Thus all the conditions of the classical verification theorems (e.g., Krylov (1980, Theorem 1.4.5), Fleming and Rishel (1975, Theorem VI.4.2)) are satisfied, and we

may conclude that f^* is in fact optimal. Alternatively, we may appeal to the martingale optimality principle by noting that under policy f^* , the value function $V(x)$ of (34) is a martingale for the optimally controlled process, $\{X_t^*\}$. This of course follows from the fact that from the previous corollary, we have

$$E(\exp\{-\eta^+(X_{t+s}^* - X_t^*)\} | X_t^*) = \exp\{s \cdot Q(\eta^+)\} \equiv 1$$

since $Q(\eta^+) \equiv 0$. \square

REMARK 3. Note that since the drift of the Brownian motion is positive, the expected hitting time of the lower barrier, a , is infinite. Thus, under the policy given above, the expected time to ruin is infinite, since the time to ruin is a defective probability for a Brownian motion with positive drift. In fact, it is clear to see that any policy which determines a positive drift for the wealth process will also have infinite expected time to ruin. This makes a comparison of policies based on the performance measure "expected time to ruin" vacuous. In §6 however, we will show that this strategy does *maximize* the time until the lower barrier is hit, in a particular stochastic ordering.

REMARK 4. The reader may have noticed by now that the policy described above is equivalent to the one obtained by maximizing the ratio of the drift function over the diffusion function for the diffusion describing the wealth process. Specifically, from equations (5) and (6), we see that under an arbitrary policy f , the wealth process is a diffusion with drift coefficient function $\mu(f) = f\mu + \alpha$, and with diffusion coefficient function $\sigma^2(f) = f^2\sigma^2 + \beta^2 + 2\rho\sigma\beta f$. Optimizing the ratio $A(f) \equiv \mu(f)/\sigma^2(f)$ will result in $f^* = C$. In discrete-time gambling problems for an *ordinary* investor gambling on a sequence of random variables with mean μ and variance σ^2 , the ratio $I = -2\mu/\sigma^2$ has been called the "inequity" by Dubins and Savage (1965, 1976). Our results therefore indicate that the optimal policy should minimize the inequity. This raises the question as to what should the optimal policy be for a more general controlled diffusion. The answer was given in the fundamental paper by Pestien and Sudderth (1985), who proved that in fact the optimal policy is to maximize the ratio A . They of course did not deal with our specific model, and in fact for the examples they studied were not able to make use of the direct application of the HJB equations we employ here, which allows us to obtain the value function explicitly. In particular, one of the examples they considered, "continuous-time red and black," corresponds to our problem with $\alpha = \beta = \rho = 0$, and for this problem the HJB methods employed here fail, since (31) shows that $Q(\eta) \equiv -\mu^2/2\sigma^2$, and therefore the requirement $Q(\eta) = 0$ is vacuous. However, we have obtained our results from first principles without having to apply Pestien and Sudderth (1985), and furthermore, our approach allowed us to exhibit clearly the relation between minimizing ruin and exponential utility, in that the complicated nonlinear differential equations resulting from the HJB equations admit exponential solutions for these cases. We will use the results of Pestien and Sudderth (1985) later in §7 to show that the conjecture of Ferguson (1965) does not hold for the case where there is positive interest rate.

REMARK 5. It is very interesting to observe that the constant C depends on the drift and diffusion coefficients of the individual Brownian motions *only through their respective ratios*. In particular, if we let $m \equiv \alpha/\mu$, and $v \equiv \beta/\sigma$, then some simple algebraic manipulations will show that in fact we may write the constant C as

$$C = \sqrt{m^2 + v^2 - 2\rho vm} - m.$$

5. Minimizing probability of ruin: The constrained case. In this section we consider again the problem of minimizing the probability of ruin. However we suppose now that there is no possibility of borrowing, so that $0 \leq f_t \leq X_t^f$.

The solution to the unconstrained problem treated earlier shows that we need consider only two cases, namely (1) the case $C < 0$, and (2) the case $C > 0$. It is clear that when $C < 0$, the unconstrained optimal policy never becomes feasible for the constrained case, and so therefore the constrained optimal policy is to never invest, i.e., $f^* = 0$. A more interesting scenario develops when $C > 0$. Clearly, if $C < a \leq x \leq b$, then $f^* = C$, so the interesting case is where $a < C$.

Without loss of generality, suppose that $a < C < b$, where

$$(38) \quad C = \frac{\mu}{\sigma^2 \eta^+} - \frac{\rho \beta}{\sigma}.$$

It follows from the previous sections that for $C \leq x \leq b$, the optimal policy invests C , i.e., $f^*(x) = C$, for $x > C$. There is a breakpoint at C in this case, and the only question is what is the optimal policy for $x < C$. As we will show below, in this case we have a hybrid policy, in that we invest "boldly" below C .

For reasons that will become clear soon, define the following functions:

$$(39) \quad H(x) \equiv \frac{1}{\sqrt{\sigma^2 \beta^2 (1 - \rho^2)}} \arctan \left(\frac{\sigma^2 x + \rho \sigma \beta}{\sqrt{\sigma^2 \beta^2 (1 - \rho^2)}} \right),$$

$$(40) \quad G(x) \equiv [H'(x)]^{\mu/\sigma^2} \exp \left\{ -2 \left(\alpha - \frac{\mu \rho \beta}{\sigma} \right) H(x) \right\},$$

$$(41) \quad K(x) \equiv \left[\frac{G(x)}{\eta^+} (1 - e^{-\eta^+(b-x)}) + \int_a^x G(u) \, du \right]^{-1}.$$

Note that $H'(x) = [\sigma^2 x^2 + \beta^2 + 2\rho\sigma\beta x]^{-1}$.

THEOREM 3. *Suppose $a < C < b$, and $a < x < b$, then the optimal policy to maximize the probability of hitting the level b before hitting the level a , with no borrowing allowed is*

$$(42) \quad f^*(x) = \min\{x, C\} = \begin{cases} x & \text{for } a \leq x \leq C, \\ C & \text{for } C \leq x \leq b, \end{cases}$$

and the optimal value function is

$$(43) \quad V(x) = \begin{cases} K(C) \int_a^x G(u) \, du & \text{for } a \leq x \leq C, \\ 1 + K(C) \frac{G(C)}{\eta^+} (e^{-\eta^+ b} - e^{-\eta^+ x}) & \text{for } C \leq x \leq b. \end{cases}$$

PROOF. For this case, the HJB equations are

$$(44) \quad 0 = \sup_{0 \leq f \leq x} \mathcal{A}^f V(x)$$

and the boundary conditions remain $V(a) = 0, V(b) = 1$. For $x \geq C$, then the uncon-

strained optimal policy is feasible, and so clearly, in that region we have $f^*(x) = C$, and furthermore, in that region, the HJB equation satisfies

$$(45) \quad 0 = \frac{1}{2}\beta^2(1 - \rho^2)V_{xx} - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\frac{V_x^2}{V_{xx}} + \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)V_x, \quad \text{for } C \leq x \leq b,$$

with the boundary condition $V(b) = 1$.

The only question then is what to do in the region $a \leq x < C$. We will “guess” that the constrained optimal policy invests as much as it can in that case, which in this case is simply x , so that $f^*(x) = \min\{x, C\}$. If this is true, then clearly, in this region the optimal value function must satisfy

$$(46) \quad 0 = (x\mu + \alpha)V_x + \frac{1}{2}(\sigma^2x^2 + \beta^2 + 2\rho\sigma\beta x)V_{xx}, \quad \text{for } a \leq x < C,$$

subject to a boundary condition $V(a) = 0$. But since we require that the value function be twice continuously differentiable, i.e., $V(x) \in \mathcal{C}^2$, we also must have the “smooth pasting” conditions described by Krylov (1980, p. 32) (these are called “*matching* and *optimality* conditions on the boundary,” by Whittle (1983, Chapter 37), and the “high contact” principle by others).

This principle dictates that the optimal value function be continuous and twice differentiable at the boundary, C . To make this rigorous, we require

$$(47) \quad V(C-) = V(C+),$$

$$(48) \quad V_x(C-) = V_x(C+),$$

$$(49) \quad V_{xx}(C-) = V_{xx}(C+).$$

Our “guess” will be verified if we can find a value function that satisfies the differential equations (45) and (46) as well as the boundary conditions and all the smooth pasting conditions.

Now, from the development in the previous section, we know that the solution to the nonlinear ordinary differential equation in (45) that satisfies $V_{xx} < 0$ is

$$(50) \quad V(x) = \kappa - \frac{\delta}{\eta^+}e^{-\eta^+x},$$

where κ and δ are constants to be determined from the boundary conditions at C and b . The boundary condition $V(b) = 1$ determines the constant κ as

$$\kappa = 1 + \frac{\delta}{\eta^+}e^{-\eta^+b}$$

and thus, to the right of the breakpoint C , we have

$$(51) \quad V(x) = 1 + \frac{\delta}{\eta^+}(e^{-\eta^+b} - e^{-\eta^+x}), \quad \text{for } C \leq x \leq b,$$

where it remains to determine the constant δ .

For $\rho^2 \neq 1$, the quadratic term in (46) is always greater than 0, so we must solve the linear second order ordinary differential equation

$$\frac{V_{xx}}{V_x} = -\frac{2(x\mu + \alpha)}{\sigma^2x^2 + \beta^2 + 2\rho\sigma\beta x}.$$

Integrating twice yields the solution

$$V(x) = c_1 \int_a^x G(u) du + c_2,$$

where the function $G(\cdot)$ was defined earlier in (40), and the two constants of integration, c_1, c_2 , must be determined from the appropriate boundary and smooth pasting conditions.

The boundary condition $V(a) = 0$ determines that $c_2 = 0$, thus for the appropriate constant c_1 , to the left of the breakpoint C , we have

$$(52) \quad V(x) = c_1 \int_a^x G(u) du.$$

To determine the two constants c_1 and δ , we must now make use of the smooth pasting conditions. Differentiating (51) and (52) appropriately show that the three conditions (47), (48) and (49) become respectively

$$(53) \quad c_1 \int_a^C G(u) du = 1 + \frac{\delta}{\eta^+} (e^{-\eta^+ b} - e^{-\eta^+ C}),$$

$$(54) \quad c_1 G(C) = \delta e^{-\eta^+ C},$$

$$(55) \quad c_1 G'(C) = -\eta^+ \delta e^{-\eta^+ C}.$$

Condition (54) determines δ as

$$\delta = c_1 e^{\eta^+ C} G(C),$$

and when this is placed back into (53), this determines the constant c_1 as

$$c_1 = K(C)$$

where the function $K(\cdot)$ was defined earlier in (41).

Substituting these constants back into (51) and (52) yield the value function $V(x)$ given by (43).

It remains to verify condition (55). To that end, differentiate (40) to find that

$$(56) \quad G'(x) = e^{-2(\alpha - \rho\beta\mu/\sigma)H(x)} \left[\frac{\mu}{\sigma^2} H'(x)^{\mu/\sigma^2 - 1} H''(x) - 2 \left(\alpha - \frac{\rho\beta\mu}{\sigma} \right) H'(x)^{\mu/\sigma^2 + 1} \right].$$

Using the fact that $H'(x) = [\sigma^2 x^2 + \beta^2 + 2\rho\sigma\beta x]^{-1}$, and therefore $H''(x) = -2(\sigma^2 x + \rho\sigma\beta)H'(x)^2$, in (56) and simplifying shows that we may write

$$(57) \quad G'(x) = -2(\alpha + \mu x)H'(x)G(x).$$

Evaluating (57) at $x = C$, and substituting into condition (55), and canceling the like terms from both sides therefore shows that we require

$$(58) \quad \eta^+ = 2(\alpha + \mu C)H'(C).$$

Substituting (23) for C in (58) shows, after some tedious but straightforward manipulations, that it is equivalent to $Q(\eta^+) = 0$, which defined η^+ in (31). Thus, all the conditions are satisfied, and so the theorem is proved. \square

REMARK 6. Theorem 3 could also have been proved via the theorem in Pestien and Sudderth (1985), although the approach taken here yields the optimal value function in explicit form. In fact, the results of Pestien and Sudderth (1985) imply that a more general result holds. Namely, if $C > 0$, and the constraint on investment is given by a nonnegative Borel function ψ , where $f \leq \psi(X_f^t)$, for all f , then the optimal policy is $f^*(x) = \min\{\psi(x), C\}$.

6. Minimizing expected discounted penalty of ruin. In this section, we return to the case where there are no constraints on the control. Suppose however that there is a discount rate λ , and that there is a large (constant) penalty M that must be paid if and when the lower barrier a is hit. The present value of the amount due upon hitting this barrier is therefore $Me^{-\lambda\tau_d^t}$. This is of particular interest in the pension fund management application.

Our objection in this section is to determine a strategy that *minimizes the expected discounted penalty paid*, which is clearly the same strategy that minimizes $E(e^{-\lambda\tau_d^t}|X_0 = x)$, where $a < x$. We will allow an unlimited amount of borrowing (i.e., f is unconstrained).

Define the constants q^+ , Δ , Γ by

$$(59) \quad q^+ \equiv \frac{(\alpha - \rho\beta\mu/\sigma) + \sqrt{\Delta}}{\beta^2(1 - \rho^2)},$$

$$(60) \quad \Delta \equiv \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)^2 + \beta^2(1 - \rho^2)\left(\frac{\mu}{\sigma}\right)^2 + 2\lambda\beta^2(1 - \rho^2),$$

$$(61) \quad \Gamma \equiv \frac{\mu}{\sigma^2 q^+} - \frac{\rho\beta}{\sigma},$$

and note that $q^+ \geq 0$.

For this case once again, the optimal policy invests a fixed constant.

THEOREM 4. Let $F(x) = \inf_f E(e^{-\lambda\tau_d^t}|X_0 = x)$, with f^* denoting the optimal policy, then

$$(62) \quad f^*(x) = \Gamma, \text{ for all } x > a$$

and the optimal value function is

$$(63) \quad F(x) = e^{-q^+(x-a)}.$$

PROOF. For this problem the HJB equation becomes (see, e.g., Krylov (1980, Theorem 1.4.5))

$$(64) \quad 0 = \inf_f \mathcal{A}^f F(x) - \lambda F(x),$$

with the boundary condition $F(a) = 1$. In other words, we must solve

(65)

$$\min_f \left[-\lambda F + (f\mu + \alpha)F_x + \frac{1}{2}(\alpha^2 f^2 + \beta^2 + 2\rho\sigma\beta f)F_{xx} \right] = 0, \text{ for } a < x,$$

with the boundary condition $F(a) = 1$.

Optimizing with respect to f shows that so long as $F_x < 0, F_{xx} > 0$, the optimal control is given by

$$(66) \quad f^*(x) = -\frac{\mu}{\sigma^2} \frac{F_x}{F_{xx}} - \frac{\rho\beta}{\sigma},$$

which when placed back into (65) yields the nonlinear differential equation

$$(67) \quad \frac{1}{2}\beta^2(1 - \rho^2)F_{xx} - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \frac{F_x^2}{F_{xx}} + \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)F_x - \lambda F = 0, \text{ for } x > a.$$

Let us again try a solution of the form $F(x) = (p/q)e^{-qx}$, with $F_x = -pe^{-qx}$ and $F_{xx} = qpe^{-qx}$. Substituting into (67), we see that we require

$$(68) \quad \frac{1}{2}\beta^2(1 - \rho^2)q^2 - \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)q - \left(\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 + \lambda\right) = 0,$$

and we will denote the positive and negative roots to this by q^+, q^- , respectively. The boundary condition $F(a) = 1$ determines the constant p as $p = qe^{qa}$, so that $F(x) = e^{-q(x-a)}$, for $x > a$. Since we require that $F_x < 0$, and $F_{xx} > 0$ for a true minimum, clearly it is the positive root, q^+ that we are interested in, so finally

$$(69) \quad F(x) = e^{-q^+(x-a)}.$$

When this is placed back into the optimal control (66), we find that

$$(70) \quad f^* = \frac{\mu}{\sigma^2 q^+} - \frac{\rho\beta}{\sigma},$$

so we see that the optimal control always invests the *fixed* constant Γ .

Verification via the martingale optimality principle is similar in this case to that of §4, since the optimal wealth process is again equivalent to a Brownian motion. In particular,

$$(71) \quad dX_t^{f^*} = \left[\frac{\mu^2}{\sigma^2 q^+} + \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right) \right] dt + \sqrt{\left(\frac{\mu}{\sigma q^+}\right)^2 + \beta^2(1 - \rho^2)} dW_t,$$

from which it is clear to see that the process $\{e^{-\lambda t} F(X_t^{f^*})\}$ is a martingale, where $F(x)$ is given by (69). \square

REMARK 7. It is interesting to observe that we can interpret $F(x)$ as the Laplace transform of the hitting time to a from level x , and so for any $\lambda \geq 0$ we have obtained the strategy that *minimizes* this Laplace transform, which is one of the many stochastic orderings discussed in Stoyan (1983). However, as noted before, when we let $\lambda \downarrow 0$, we do not have $\lim_{\lambda \rightarrow 0} F(x) = 1$, as under the optimal policy, τ_a^* is a defective random variable. Theorem 4 above is somewhat related to Theorem 2.1 of Orey et al. (1987), which was established by alternative means.

7. Positive interest rate. In this section we consider the case where there is a positive interest rate $r > 0$. In particular, suppose that beside the risky stock, whose price at time t is given by P_t , there is also a bond, whose price at time t , B_t , evolves as $dB_t = rB_t dt$. In this case, any wealth not invested in the stock, $X_t^f - f_t$, will be held in the bond. Since $dP_t = P_t \cdot dZ_t$, we see that for any policy f , the wealth process evolves as

$$(72) \quad dX_t^f = f_t dZ_t + r(X_t^f - f_t) dt + dY_t \\ = [rX_t^f + f_t(\mu - r) + \alpha] dt + f_t \sigma dW_t^{(1)} + \beta dW_t^{(2)}.$$

The generator of the wealth process in this case is

$$(73) \quad \mathcal{A}_r^f g(t, x) = g_t + [f(\mu - r) + rx + \alpha] g_x + \frac{1}{2} [f^2 \sigma^2 + \beta^2 + 2\rho\sigma\beta f] g_{xx} \\ \equiv \mathcal{A}^f g(t, x) + r(x - f) g_x,$$

where \mathcal{A}^f is the generator of (7), without an interest rate.

7.1. Exponential utility. We can extend the results of §3 regarding maximizing exponential utility to this case with only minor modifications. Specifically, for the problem of maximizing utility from terminal wealth, the HJB equations in this case become (for $t < T$),

$$(74) \quad \sup_f \{ \mathcal{A}_r^f V(t, x) \} = 0, \quad V(T, x) = u(x),$$

where again we have $V(t, x) = \sup_f E^{t,x}[u(X_T^f)]$. Once again, for each (t, x) , we must solve the nonlinear partial differential equation of (74), and then find the value $f_t(x)$ which maximizes the function

$$(75) \quad V_t + [rx + f(\mu - r) + \alpha] V_x + \frac{1}{2} [f^2 \sigma^2 + \beta^2 + 2\rho\sigma\beta f] V_{xx}.$$

Assuming that the HJB equation of (74) has a classical solution V , which satisfies $V_x > 0$, $V_{xx} < 0$, we differentiate with respect to f in (75) to find the optimizer

$$(76) \quad f_t^* = -\frac{\mu - r}{\sigma^2} \left(\frac{V_x}{V_{xx}} \right) - \frac{\rho\beta}{\sigma}.$$

When this is placed back into (75), the HJB equation (74) becomes, after some simplification, equivalent to the following nonlinear Cauchy problem for the value function V :

$$(77) \quad V_t + \left[rx + \alpha - \frac{\rho\beta}{\sigma} (\mu - r) \right] V_x - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \frac{V_x^2}{V_{xx}} + \frac{1}{2} \beta^2 (1 - \rho^2) V_{xx} = 0, \quad \text{for } t < T \\ (78) \quad V(T, x) = u(x).$$

We are concerned here with the case where the investor has an exponential utility function, namely the function given by (8). Pliska (1986) solved the optimal investment problem for this case for an ordinary investor without the random risk process

Y_t , i.e., when $\alpha = \beta = 0$. He found that the value function in that case was

$$\lambda - \frac{\gamma}{\theta} \exp \left\{ -\theta x e^{r(T-t)} - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 (T - t) \right\}$$

and the optimal strategy was to invest $((\mu - r)/\theta\sigma^2)e^{-r(T-t)}$ in the risky stock. Using his results as a first guess therefore, to solve the partial differential equation in (77), try to fit a solution of the form

$$(79) \quad V(t, x) = \lambda - \frac{\gamma}{\theta} \exp \left\{ -\theta x e^{r(T-t)} - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 (T - t) + h(T - t) \right\},$$

where $h(\cdot)$ is a suitable function. The boundary condition $V(T, x) = \lambda - (\gamma/\theta)e^{-\theta x}$, implies that $h(0) = 0$. Inserting this trial solution (79) into (77) shows that we require

$$h'(T - t) = -\theta \left(\alpha - \frac{\rho\beta(\mu - r)}{\sigma} \right) e^{r(T-t)} + \theta^2 \frac{1}{2} \beta^2 (1 - \rho^2) e^{2r(T-t)}.$$

Integrating and setting $h(0) = 0$ then shows that

$$(80) \quad h(T - t) = -\theta \left(\alpha - \frac{\rho\beta(\mu - r)}{\sigma} \right) \left(\frac{e^{r(T-t)} - 1}{r} \right) + \theta^2 \frac{1}{2} \beta^2 (1 - \rho^2) \left(\frac{e^{2r(T-t)} - 1}{2r} \right).$$

Noting that the function in (79) satisfies

$$(81) \quad V_x(t, x) = [V(t, x) - \lambda] \cdot [-\theta e^{r(T-t)}],$$

$$(82) \quad V_{xx}(t, x) = [V(t, x) - \lambda] \cdot [\theta^2 e^{2r(T-t)}],$$

we can now substitute the values for V_x and V_{xx} from equations (81) and (82) into (76) to find the optimal control, which in this case is

$$f_t^*(x) = \frac{\mu - r}{\theta\sigma^2} e^{-r(T-t)} - \frac{\rho\beta}{\sigma}.$$

REMARK 8. Note that under this policy, the amount invested in the risky stock is no longer constant, although it is independent of the wealth level. It is interesting to observe that this policy invests more as the deadline gets closer.

When this control is placed back the evolutionary equation (72), we find that the optimal wealth process is equivalent to the process that satisfies the stochastic differential equation

$$(83) \quad dX_t^* = [rX_t^* + \gamma_t + \theta\gamma_2 e^{-r(T-t)}] dt + \sqrt{\gamma_2 e^{-2r(T-t)} + \gamma_3} dW_t,$$

where $\gamma_1 = \alpha - (\rho\beta/\sigma)(\mu - r)$, $\gamma_2 = ((\mu - r)/\theta\sigma)^2$ and $\gamma_3 = \beta^2(1 - \rho^2)$, and W_t is a standard Brownian motion.

Since this is linear equation, standard results (e.g., Karatzas and Shreve (1988, p. 354)) tell us that it admits the unique strong solution

$$(84) \quad X_t^* = X_0 e^{rt} + \frac{\gamma_1}{r} (e^{rt} - 1) + \theta \gamma_2 t e^{-r(T-t)} \\ + e^{rt} \int_0^t \sqrt{\gamma_2 e^{-2rT} - \gamma_3 e^{-2rs}} dW_s, \quad \text{for } t < T.$$

This, together with the fact that the value function is twice-continuously differentiable, suggests the following.

THEOREM 5. *When there is a positive interest rate r , and the objective is to maximize utility from terminal wealth, at the fixed terminal time T , for the utility function*

$$\lambda - \frac{\gamma}{\theta} e^{-\theta x}$$

the optimal value function is given by

$$(85) \quad V(t, x) = \lambda - \frac{\gamma}{\theta} \exp \left\{ -\theta x e^{r(T-t)} - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 (T-t) + h(T-t) \right\},$$

where the function $h(T-t)$ is given by (80), and the optimal policy is to invest

$$(86) \quad f_t^* = \frac{\mu - r}{\theta \sigma^2} e^{-r(T-t)} - \frac{\rho \beta}{\sigma}$$

in the risky stock at time t .

It is important to note that the optimal control given above reduces to the optimal control for the simple case of a complete market ($\alpha = \beta = 0$), as described in Pliska (1986), plus another constant, $-\rho\beta/\sigma$, which is independent of the rate at which external funds accrue to the firm (i.e., α).

It is an open question whether similar decompositions arise for other utility functions in this incomplete market.

PROOF. While it is not hard to check that all the conditions of the verification theorems cited previously are met, and therefore the control given in (86) is in fact optimal, it is instructive to verify the optimality by use of the martingale optimality principle. Since an application of Ito's formula shows that $\{V(t, X_t^i) - \lambda\}$ is a supermartingale under any admissible policy f , the theorem will be completely proved if we can prove the following

LEMMA 1. *Under the control policy (86), the process $\{V(t, X_t^*) - \lambda: t \leq T\}$, where $V(t, x)$ is given by (85), is a martingale.*

PROOF. Apply Ito's formula (Karatzas and Shreve (1988, Theorem 3.3.6)) to the function $V(t, x) - \lambda$ to get (recall (83)),

$$(87) \quad d[V(t, X_t^*) - \lambda] = (V_t + [rX_t^* + \gamma_1 + \theta \gamma_2 e^{-r(T-t)}] V_x \\ + \frac{1}{2} [\gamma_3 + \gamma_2 e^{-2r(T-t)}] V_{xx}) dt \\ + \left[\sqrt{\gamma_3 + \gamma_2 e^{-2r(T-t)}} \right] V_x dW_t$$

where the constants $\gamma_1, \gamma_2, \gamma_3$ were defined previously.

Noting now that

(88)

$$V_t(t, x) = [V(t, x) - \lambda] \cdot [rx\theta e^{r(T-t)} + \frac{1}{2}\theta^2\gamma_2 + \theta\gamma_1 e^{r(T-t)} - \frac{1}{2}\theta^2\gamma_3 e^{2r(T-t)}]$$

substitute this, as well as equations (81) and (82), back into (87), and simplify to get

(89)
$$d[V(t, X_t^*) - \lambda] = [V(t, X_t^*) - \lambda] \cdot [-\theta\sqrt{\gamma_2 + \gamma_3 e^{2r(T-t)}} dW_t].$$

This is a linear stochastic differential equation with the solution (see, e.g., Karatzas and Shreve (1988, equation (5.6.34))),

(90)

$$\frac{[V(t, X_t^*) - \lambda]}{[V(0, X_0^*) - \lambda]} = \exp\left\{-\frac{\theta^2}{2} \int_0^t (\gamma_2 + \gamma_3 e^{2r(T-u)}) du - \theta \int_0^t \sqrt{\gamma_2 + \gamma_3 e^{2r(T-u)}} dW_u\right\}.$$

It is now easy to recognize the right-hand side of (90) as the familiar *exponential martingale* (e.g., Karatzas and Shreve (1988, Proposition 3.5.12)), and since the Novikov condition holds, we have proved that for $t \leq T$, the process $\{V(t, X_t^*) - \lambda\}$ is in fact a martingale. By the martingale optimality principle, this suffices to prove the theorem. \square

REMARK 9. The probability that terminal wealth for an exponential utility-maximizer is positive is relatively straightforward to obtain. In particular, for a constant X_0 , and $s \leq t < T$, X_t^* , as given by equations (83) and (84), is a Gaussian process with mean function $m(t) = EX_t^*$, and covariance function $\rho(s, t) = E[(X_s^* - m(s))(X_t^* - m(t))]$, where

(91)
$$m(t) = X_0 e^{rt} + \frac{\gamma_1}{r} (e^{rt} - 1) + \theta\gamma_2 t e^{-r(T-t)},$$

(92)
$$\rho(s, t) = s e^{-r(T-t)} e^{-r(T-s)} + \frac{\gamma_3}{2r} (e^{r(t+s)} - e^{r(t-s)}).$$

The variance function is $\nu(t) \equiv \rho(t, t)$, so

(93)
$$\nu(t) = e^{2rt} \left(t e^{-2rT} + \frac{\gamma_3}{2r} \right) - \frac{\gamma_3}{2r}.$$

Therefore, the probability that terminal wealth is positive is simply $P(X_T^* > 0) = 1 - \Phi(-\zeta_T)$, where $\zeta_T = m(T) / \sqrt{\nu(T)}$, and Φ is the cdf of the standard normal. This of course, is *not* the probability that ruin did not occur by time T , i.e., $P(X_T^* > 0) \neq P(\inf_{0 \leq t \leq T} X_t^* > 0)$.

7.2. *Maximizing survival or minimizing ruin.* While it is possible to solve the HJB equations directly to obtain the optimal value function for the problem of *minimizing the probability of ruin* for the case where $r > 0$, the resulting solutions are quite complicated and the manipulations needed to obtain the optimal policy very cumbersome. In this case, it is much easier to apply the results of Pestien and Sudderth (1985) to obtain the optimal policy directly. Specifically, note that when there is an interest rate, $r > 0$, the wealth process evolves as a diffusion with infinitesimal drift $\mu(f) = rx + f(\mu - r) + \alpha$, and diffusion parameter $\sigma^2(f) = f^2\sigma^2 + \beta^2 + 2\rho\sigma\beta f$. Choosing f^* to maximize the ratio $\mu(f)/\sigma^2(f)$, yields the following.

THEOREM 6. *The optimal control to maximize $P_x(X_{\tau_f}^f \geq b)$ (and hence to minimize the probability of ruin) is the state-dependent function*

(94)

$$f^*(x) = \frac{1}{\mu - r} \left[\sqrt{\left(rx + \alpha - \frac{\rho\beta(\mu - r)}{\sigma} \right)^2 + (1 - \rho^2)\beta^2 \left(\frac{\mu - r}{\sigma} \right)^2} - (rx + \alpha) \right].$$

REMARK 10. Clearly, this state-dependent policy is not equivalent to the policy given by (86), for any value of the risk aversion parameter θ , and thus it is immediate that the conjecture of Ferguson (1965) does not hold for this case.

It is interesting to note that the control given in (94) is *decreasing* in the wealth x , with $f^*(0) = C$, and $\lim_{x \rightarrow \infty} f^*(x) = 0$. This agrees with simple intuition, of course, in that if the objective is to minimize the probability of ruin, then when there is a positive interest rate, the optimal policy will invest less in the risky stock (and hence more in the risk-free bond) than would be the case if there was no interest rate.

The complicated structure of the optimal policy given in (94) demonstrates why it is rather difficult to employ HJB techniques for this problem in this case, since the HJB approach is contingent upon an explicit evaluation of the value function, which in turn will yield the optimal control. However, when we substitute the control in (94) back into the drift and diffusion functions for the wealth process, it is apparent that it is quite difficult to obtain the value function, or equivalently, the ruin probability of the optimal process, in a simple manageable form (although it can obviously be expressed as a complicated definite integral).

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