Drawdown Minimization

Drawdown versus Shortfall

One measure of riskiness of an investment is "drawdown", defined, most often in the asset management space, as the decline in net asset value from a historic high point. Mathematically, if the net asset value is denoted by $V_t, t \geq 0$, then the current "peak-to-trough" drawdown is given by $D_t = V_t - \max_{0 \leq u \leq t} V_u$. The maximum drawdown, $\max_{0 \leq u \leq t} D_u$, is a statistic that the CFTC forces managed futures advisors to disclose, and so many investment advisors and managers implicitly face drawdown constraints in setting their investment strategies. Hedge funds, for example, implicitly face drawdown constraints in that many multiperiod hedge fund contracts reflect investor preferences related to the maximum drop in a fund's asset value from the previous peak. These often include a high-water-mark provision that sets the strike price of each period's incentive fee equal to the all-time high of fund value (*see* **Hedge Funds**).

Another measure of riskiness that is related but often confused terminologically with drawdown is that of "shortfall", which is simply the gap, or loss level of the current value from the initial or some other given value. This value could be constant but more often is determined by a stochastic exogenous or endogenous benchmark. For example, the shortfall with respect to the endogenous benchmark of the running maximum is the drawdown.

Since drawdown and shortfall are essentially equivalent in single period models, the research on the topic reviewed in this article is focused on multiperiod and, in particular, on continuous-time models pioneered in [16] and [17] where optimal portfolio rules are derived by solving a multiperiod portfolio optimization problem. The minimization of short-fall probability in a single-period model dates back to [18] and [21]. *See* **Value-at-Risk**; **Expected Shortfall** for work on portfolio selection with drawdown constraints in single period mean-variance models.

In the continuous time framework, there is an implicit nonnegativity constraint on wealth, which is one form of a shortfall constraint (*see* **Merton Problem**).

The models reviewed here differ in their assumptions regarding investment horizons (finite or infinite), constraints (fixed or stochastic benchmark), stochastic processes (diffusion with and without jumps), as well as objective function (purely probabilistic or expected utility based). In general, without transactions costs, the incorporation of drawdown constraints induces a portfolio insurance strategy: specifically, in the stationary stochastic model case, the strategy is that of a constant proportions portfolio insurance (CPPI) with different "floor" levels determined by the horizon and the objective (*see* **Transaction Costs** for portfolio optimization with transaction costs).

In this case, the risky asset price is assumed to follow a geometric Brownian motion with drift $\mu + r$, and diffusion coefficient σ^2 where r is the rate of return on cash. Hence, as is standard (*see* **Merton Problem**), the dynamics of the investor's wealth portfolio are given by

$$\mathrm{d}W_t = rW_t\mathrm{d}t + x_t[\mu\mathrm{d}t + \sigma\mathrm{d}Z_t] \tag{1}$$

where Z_t is a standard Brownian motion and where x_t denotes the dollar holdings of stock.

For reference, investment strategies that are of the form

$$x_t = k \times W_t \tag{2}$$

are called a *constant proportion rebalanced portfolio rule* (*see* **Fixed Mix Strategy**), and are optimal in a variety of settings ([7]). Investment strategies that are of the form

$$x_t = k(W_t - F_t) \tag{3}$$

are referred to as *constant proportion portfolio insurance strategies (CPPI)* with scale multiple k and floor F_t (*see* **Constant Proportion Portfolio Insurance**).

Such CPPI strategies are, in fact, constant proportional strategies on the *surplus* $W_t - F_t$ and in a pure diffusion setting insure that wealth remains above the floor level F_t at all times, (although it is possible that F_t serves as an absorbing barrier). These strategies effectively synthesize an overlay of a put option on top of the wealth generated by a constant proportional strategy, and are at the core of many of the strategies that have been discussed here.

Infinite Horizon Drawdown Minimization

In a seminal paper on the subject, Grossman and Zhou [15] show how to extend the Merton framework to encompass a more general drawdown constraint of $W_t \geq \alpha M_t$, where W_t is the wealth level at time t, M_t is the running maximum wealth up to that point, that is, $M_t = \max_{0 \leq u \leq t} W_u$, and α is an exogenous number between 0 and 1.

The motivation for this constraint is that, in practice, a fund manager may implicitly be subject to redemptions that depend on whether the manager's portfolio stays above a (possibly discounted) previous high, M_t.

For an investor with constant relative risk aversion utility, Grossman and Zhou [15] show that the optimal policy implies an investment in the risky asset at time t in proportion to the "surplus" $W_t - \alpha M_t$, that is, a CPPI ([3]) strategy with floor given by a multiple of the running maximum wealth, $F_t = \alpha M_t$. The analysis of Grossman and Zhou [15] is extended in [12] by allowing for intermediate consumption.

Infinite Horizon Shortfall Minimization

There are a variety of approaches and objectives related to shortfall minimization. For example, in a stochastic model where rational investment strategies may enable wealth to hit a given shortfall (e.g., liability driven models), one might choose a strategy that minimizes the *probability* of ever hitting this shortfall level directly ([5, 6, 8, 9, 11, 13, 19, 20]), or one might incorporate a (expected) *shortfall* constraint into other objectives such as a standard utility maximization framework ([1, 2, 4, 20] and [22]).

Directly minimizing the probability of hitting a shortfall level is a relevant objective only in economic settings where there is a possibility that a shortfall level is reached under a rational investment strategy. One such setting is the case of external risk, such as an insurance company, or liabilities as treated in [5], where the investor's total wealth evolves now according to a combination of an uncontrolled Brownian motion (the "risk" part), and a controlled geometric Brownian motion (the investment possibility). Drawdown and shortfall prevention strategies for deterministic liabilities are treated in [6] where it is shown that if the initial wealth or reserve is below the funding level of the perpetual liability, the optimal strategy is linear in the negative surplus, that is, has an *inverse* CPPI structure, namely $k(F - W)$. For initial wealth above the funding level, various CPPI strategies using the funding level as the floor are optimal for a variety of utility and probabilistic objectives.

Other settings of interest include cases where there is an exogenous and uncontrollable benchmark relative to which the shortfall is measured. Infinite horizon probabilistic objectives are treated in [8] in an incomplete market setting, and connects these results with those obtained from standard utility maximization problems. A risk-constrained problem that yields a CPPI related strategy is treated in [11]. Stutzer [19] treats long-horizon shortfalls and deviations from benchmarks in the context of a large-deviation approach.

Finite Horizon Shortfall Minimization

The structure of the optimal strategy changes significantly when the horizon is finite in that the optimal strategies become *replicating strategies* for various structures of finite term options. Specifically, the strategy that minimizes the *shortfall probability* starting from a wealth process below the target level is given by the replicating strategy for a digital or binary put option on the shortfall level. This is discussed in [9, 10] and in an equivalent hedging framework in [13]. The optimal dynamic policy in the case of multiple risky assets has a time-dependent component (determined by the risk premium and remaining time) and a state-dependent component, which is a function of the current percentage of the distance to the target. The minimization is treated in [13] in the context of determining a partial hedging strategy on a contingent claim that minimizes the hedging cost for a given shortfall probability. This strategy may be considered a dynamic version of the static Value-at-Risk (VaR) concept and the authors label it *quantile hedging*. The potential riskiness of such a strategy is illustrated in [9, 10] via the fact that since it replicates a digital or binary option, the strategy effectively acts as the delta of the digital option with all the instability of that delta as the term decays if the strike remains unachieved. Strategies that mitigate this fact and therefore minimize the *expected shortfall* are constructed in [14]. These strategies effectively replicate options with standard put payoffs as opposed to digitals or binaries.

Utility Maximization Approaches to the Expected Shortfall

Utility maximization approaches to the expected shortfall problem also lead to optimal strategies that have optionlike features. Basak and Shapiro [1] consider an agent's utility maximization problem in a model with a VaR constraint, which states that the probability of his wealth falling below some "floor" $\underline{W}$ is not allowed to exceed some prespecified level α:

$$\Pr(W(T) \geq \underline{W}) \geq 1 - \alpha \tag{4}$$

and is clearly related to the objectives treated earlier by Browne, Föllmer, and Leukert [9, 13, 14] (*see* **Value-at-Risk**).

In this framework, the case $\alpha = 1$ corresponds to the standard benchmark agent that does not limit losses and $\alpha = 0$ corresponds to the portfolio insurer (or put option purchaser) who maintains his wealth above the floor in all states [3].

Basak and Shapiro [1] show that the VaR constrained agent's wealth can be expressed as either (i) the portfolio insurer solution plus a short position in binary options or (ii) the benchmark agent's solution plus an appropriate position in "corridor" options (*see* **Corridor Options**). Similar to the analysis and earlier findings mentioned above they observe that since the VaR constrained agent is only concerned with the probability (and not the magnitude) of a loss, he or she chooses to leave the worst states uninsured because they are the most expensive ones to insure against. Thus, as in [14], Basak and Shapiro [1] examine a so-called LEL-RM (limited-expected-losses-based risk management) strategy, which remedies some of the shortcomings of the VaR constrained solution. Other related papers considering variants of these results are found in [2, 22]

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Related Articles

Constant Proportion Portfolio Insurance; **Corridor Options**; **Expected Shortfall**; **Fixed Mix Strategy**; **Hedge Funds**; **Merton Problem**; **Transaction Costs**; **Value-at-Risk**.

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