

A Theory and Test of Credit Rationing: Some Further Results

By ENRIQUE R. ARZAC, ROBERT A. SCHWARTZ, AND DAVID K. WHITCOMB*

In his extension of Dwight Jaffee and Franco Modigliani's theory of credit rationing, Vernon Smith derived an important proposition about lending behavior under a fixed interest rate. Assuming risk indifference, he proved that the optimal size of the loan is proportional to the borrower's equity. This note considers whether Smith's proposition can be extended to risk-averse behavior. We show that Smith's proposition is nonextendable in a strict sense; that is, under lender risk aversion, the lender's optimal loan offer is inelastic with respect to the borrower's equity. However, in equilibrium if the interest rate is fixed but there is competition in leverage terms, the market will establish a fixed leverage ratio for all loans whose proceeds are invested in the same risk asset (whether lenders are risk neutral or risk averse).

I. Leverage Terms under Risk Aversion

Consider a lender who allocates his capital K between an investment in a riskless asset with rate of return $s-1$ and a loan of an amount L to a borrower at the fixed contractual interest rate $r-1$ ($r>s$). The borrower invests L plus equity C in a risk asset yielding the random rate of return $R-1$. The investment $C+L$ is pledged as collateral but the borrower enjoys limited liability in that any other assets he happens to own are not subject to the lender's claim.

Expressing the borrower's leverage ratio as $V=L/(C+L)$, the random return on the loan is

$$\tilde{r} = \begin{cases} r & \text{if } R \geq R^* (=rV) \\ R/V & \text{if } R < R^* \end{cases}$$

*Graduate School of Business, Columbia University, Graduate School of Business Administration, New York University, and Graduate School of Management, Rutgers University, respectively.

where R^* is the default return on the risk asset (any return below R^* is insufficient to repay the loan in full).

The lender's wealth is $W(R) = L\tilde{r} + (K-L)s$. Assuming the lender has a differentiable subjective probability distribution of returns on the risk asset, $F(R)$, with density $f(R)$ and $F(0)=0$, and a twice differentiable utility function $U(W)$, his expected utility is

$$(1) \quad E(U) = \int_0^{R^*} U[(C+L)R + (K-L)s] dF(R) + U[Lr + (K-L)s][1 - F(R^*)]$$

We now prove:

PROPOSITION 1. *At the lender's optimum, when the contractual interest rate is fixed and $C>0$:*

- (a) $dV/dC < 0$ under risk aversion;
- (b) $dV/dC = 0$ under risk neutrality.

PROOF:

(a): Differentiating (1) with respect to L yields the following necessary condition for optimal lending:

$$(2) \quad \int_0^{R^*} U'[(C+L)R + (K-L)s](R-s) dF(R) + U'[Lr + (K-L)s](r-s)[1 - F(R^*)] = 0$$

Differentiating (2) totally with respect to C yields

$$(3) \quad \int_0^{R^*} U''[W(R)](R-s)^2 \frac{dL}{dC} dF(R) + U''[W(R^*)](r-s)^2 \frac{dL}{dC} [1 - F(R^*)] + U'[W(R^*)](R^*-r)f(R^*)r \frac{dV}{dC} = 0$$

Under risk aversion, and since $R^* < r$, (3) will be satisfied iff $\text{sgn}(dL/dC) = -\text{sgn}(dV/dC)$. Then, since

$$(4) \quad \frac{dV}{dC} = \frac{V^2}{L} \left(\frac{C}{L} \frac{dL}{dC} - 1 \right)$$

dV/dC cannot be > 0 for $dL/dC < 0$. Hence, (3) will be satisfied iff $dL/dC > 0$ and $dV/dC < 0$.¹

(b): For $U''(W) = 0$, (3) immediately gives $dV/dC = 0$.

Remarks:

1) It is straightforward to verify that Proposition 1 also holds when the alternative investment available to the lender is risky.

(2) In contrast to Smith's result under risk neutrality, Proposition 1 and equation (4) show that, under risk aversion, the elasticity of the lender's optimal loan offer with respect to the borrower's equity is less than 1. Intuitively this is because an increase in C with a proportional increase in L (i.e., V constant) causes the lender to hold a riskier portfolio than he did before. With r fixed, the lender must be compensated by a decrease in leverage.

II. Leverage Terms when Costless Arbitrage by an Intermediary is Possible

If the interest rate is fixed (for example, by regulatory fiat) one would expect market competition to focus on leverage terms. In this section we allow for the possibility of arbitrage in leverage terms by an intermediary and prove the following:

PROPOSITION 2. *In a credit market operating under a fixed interest rate but otherwise unrestrained competition, a differing leverage ratio across loans is incompatible with equilibrium in the sense that it would permit the realization of arbitrage profits by an intermediary.*

¹Azzi and Cox show that $dL/dC > 0$ under risk aversion, but $dL/dC > 0$ does not by itself imply $dV/dC < 0$ without (3) as we see from (4).

PROOF:

Let the leverage ratio vary across loans, with a fixed interest rate and one risk asset that provides collateral (either directly or indirectly). Consider an intermediary who searches borrowers and lenders for leverage discrepancies. Let the intermediary lend some low V borrower L_B against collateral $C_B + L_B$ and borrow from some high V lender $L_I = L_B$ against collateral of $C_I + L_I$. Then, with $V_I > V_B$ which implies $C_I < C_B$, the intermediary need pledge no capital of his own, and is able to retain part of B 's capital in the event of default. With this transaction the intermediary obtains an absolutely preferred (i.e., stochastically dominating) change in the distribution of his final wealth as shown in Table 1.² The second inequality in the last column of Table 1 follows from $L_B = L_I$ and $R < rV_I = rL_I / (C_I + L_I)$.

Arbitrage will produce constant leverage across all loans backed up by the same kind of risk asset since (by the proof of Proposition 1), in order to obtain additional loans, intermediaries will have to offer lower leverage. To make more loans, they will have to accept higher leverage. Since the expected utility of the borrower increases with his leverage,³ the intermediary can increase the size of his loan portfolio by accepting higher leverage terms (attracting borrowers who would otherwise need to pledge more of their equity as collateral to other lenders).

²Note that this proof assumes $L_B = L_I$. The additional assumption that the intermediary can combine and/or split loans would allow one to extend the proof to the case of different loan sizes.

³Suppose the borrower invests the amount $K_B + L - M$ in the risk asset, where K_B is his initial capital and M is his investment in the riskless asset. Since, for a given leverage ratio V , $C = L(1 - V)/V$, the borrower's expected utility is

$$E(U_B) = \int_0^{R^*} U[(K_B - M - L(1 - V))R + Ms] dF(R) + \int_{R^*}^{\infty} U[(K_B + L - M)R - Lr + Ms] dF(R)$$

we immediately have that $\partial E(U_B)/\partial V > 0$.

TABLE 1

R range	Loans Defaulted	Final Wealth Generated by Intermediation
$[0, rV_B)$	L_I, L_B	$(C_B - C_I)R > 0$
$[rV_B, rV_I)$	L_I	$L_B r - (C_I + L_I)R > 0$
$[rV_I, \infty)$	none	0

ratio will vary with the riskiness of the asset backing up the loan and variations in leverage will then substitute for variations in the interest rate. However, it can readily be shown along the lines of Smith that competition in leverage terms would be insufficient to correct the inefficient allocation of claims produced by the market when the interest rate is fixed.⁴

⁴When interest rates are free to vary, our earlier article shows that competitive equilibrium implies the existence of a positive leverage structure of interest rates that is Pareto efficient.

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