Risk-Constrained Dynamic Active Portfolio Management

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Active portfolio management is concerned with objectives related to the outperformance of the return of a target benchmark portfolio. In this paper, we consider a dynamic active portfolio management problem where the objective is related to the tradeoff between the achievement of performance goals and the risk of a shortfall. Specifically, we consider an objective that relates the probability of achieving a given performance objective to the time it takes to achieve the objective. This allows a new direct quantitative analysis of the risk/return tradeoff, with risk defined directly in terms of probability of shortfall relative to the benchmark, and return defined in terms of the expected time to reach investment goals relative to the benchmark. The resulting optimal policy is a state-dependent policy that provides new insights. As a special case, our analysis includes the case where the investor wants to minimize the expected time until a given performance goal is reached subject to a constraint on the shortfall probability.
(*Portfolio Theory; Benchmarking; Active Portfolio Management; Stochastic Control*)

1. Introduction

In this paper we analyze an optimal dynamic portfolio and asset allocation policy for an investor who is concerned about the performance of a portfolio relative to the performance of a given benchmark. We take as our setting the standard continuous-time framework pioneered by Merton (1971) and others. Portfolio problems where the objective is to exceed the performance of a selected target benchmark is sometimes referred to as *active* portfolio management, whereas *passive* portfolio management just tries to *track* a benchmark, see, for example, Sharpe et al. (1995). Many professional and institutional investors in fact follow this benchmarking procedure: For example, many mutual funds take the Standard and Poors (S&P) 500 Index as a benchmark; commodity funds seek to beat the Goldman Sachs Commodity Index; bond funds try to beat the Lehman Brothers Bond Index, etc. Moreover, benchmarking is not specific to professional investors, as many ordinary investors implicitly follow a benchmarking procedure, for example, by trying to beat inflation, exchange rates, or other indices. In other applications, such as pension funds, the benchmark might be a liability. See Litterman and Winkelman (1996) for more detail on these and other benchmarks. For a treatment of active portfolio management in a static setting, see Grinold and Kahn (1995).

This paper extends the earlier analysis in Browne (1999a) of active portfolio management problems with objectives related to the achievement of relative performance goals and shortfalls (see Browne 2000 for related stochastic differential games). Specifically, Browne (1999a) considered a general problem in an incomplete market where the benchmark was only partially correlated with the active investor's investment opportunities and with investment objectives related to the achievement of investment goals and shortfalls relative to the benchmark. The specific objectives that were explicitly solved for there include:

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maximizing the probability that the investor's wealth achieves a certain performance goal relative to the benchmark before falling below it to a predetermined shortfall; minimizing the expected time to reach the performance goal; and maximizing the expected reward obtained upon reaching the goal, as well as minimizing the expected penalty paid upon falling to the shortfall level. The corresponding optimal policies obtained there are all constant proportion, or constant mix, portfolio allocation strategies, whereby the portfolio is continuously rebalanced so as to always keep a constant proportion of wealth in the various asset classes, regardless of the level of wealth. (Observe that if the proportion associated with an asset class is positive, then this rebalancing requires selling an asset when its price rises relative to the other prices, and conversely, buying the asset when its price drops relative to the others.) It is well-known that such policies have a variety of optimality properties associated with them for the ordinary portfolio problem (see, e.g., Merton 1990 or Browne 1998 for surveys) and are widely used in asset allocation practice (see Perold and Sharpe 1988 and Black and Perold 1992). Nevertheless, some investors object to using constant proportion strategies in that they dictate the same strategy for every wealth level, while their individual intuition would suggest otherwise. In this paper we address some of these issues for the complete market case where the investor is allowed to invest in all the individual components of the benchmark. From an analytic point of view, the problem addressed here is solvable only in the complete market case, which is a somewhat more restrictive setting than that of Browne (1999a). However, it is in fact the complete market case that is of most interest to active portfolio practitioners. Our analysis allows us to extend the domain of goal/shortfall-related objectives with known explicit solutions to a case that allows for a very interesting and intuitive state-dependent optimal policy. (A continuous-time active portfolio management problem with a finite-horizon probability-maximizing objective in a complete market setting was studied in Browne (1999b, 1999c). The optimal portfolio policy in that case turns out to be intimately related to hedging strategies for certain options, and as such is *both* time and state dependent.)

An outline of the remainder of the paper, and a summary of our main results are as follows: In the next section, we provide a description of the model and the problems studied. For the objectives considered here, the relevant state variable is the *ratio* of the investor's wealth to the benchmark. We then state for reference a general theorem in stochastic control for our model, which contains the specific goal-related objectives considered in the sequel as a special case. The upshot of this theorem is that it shows how the optimal value function and associated optimal control function for a general control problem can be obtained as the solution to a particular nonlinear Dirichlet problem. The theorem is a special case of the more general result in Browne (1999a), and so it is stated without proof. Because the specific goal-related problems considered in the sequel are special cases, we need only identify and then solve the appropriate nonlinear Dirichlet problem.

In §3 we apply the theorem to show that the ordinary optimal-growth portfolio policy for the case of an investor without a benchmark is once again optimal in our extended model, in that regardless of the underlying benchmark, the ordinary optimal-growth strategy will minimize the expected time until that benchmark is outperformed by any given percentage. While this result is not new, it is quite important for the sequel. In particular, it highlights the rather disturbing property of this optimal-growth policy in that it yields no insight for the portfolio manager as to how the benchmark affects the investment decision, because for this objective the benchmark is in fact irrelevant. Moreover, as we show below, not only is the policy independent of the benchmark, but the *probability* that the active manager using the optimal-growth policy reaches an investment goal before falling to a shortfall level relative to the benchmark is also *independent* of the benchmark as well as any other parameters of the assets. Given these disturbing results, we move on in §4 to consider a fractional objective that relates the time to beat the benchmark to the probability of a shortfall relative to the benchmark. For this objective, the optimal strategy is no longer a constant proportion, but rather a state-dependent amount that modulates the amount invested in the risky assets in inverse proportion to wealth. As a special case, our results are then applied to the objective of minimizing the expected time to reach the goal subject to a constraint on the shortfall probability. This objective is motivated by the interesting gambling model of Gottlieb (1985).

2. The Model

The model under consideration here consists of $k+1$ underlying processes: k (correlated) risky assets or stocks $S^{(1)},\ldots,S^{(k)}$ and a riskless asset B called a bond. The investor may invest in the risky stocks and the bond, whose price processes will be denoted, respectively, by $\{S_t^{(i)}, t\geq 0\}_{i=1}^k$ and $\{B_t, t\geq 0\}$.

The probabilistic setting is as follows: We are given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathsf{P})$, supporting k independent standard Brownian motions, $(W^{(1)},\ldots,W^{(k)})$, where $\mathcal{F}_t$ is the P-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma\{W_s^{(1)}, W_s^{(2)},\ldots,W_s^{(k)};\ 0\leq s\leq t\}$ (see e.g., Duffie 1996 for a brief review of the relevant terminology).

It is assumed that these k Brownian motions generate the prices of the k risky stocks. Specifically, following Merton (1971) and many others, we will assume that the risky stock prices are correlated geometric Brownian motions, i.e., $S_t^{(i)}$ satisfies the stochastic differential equation

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sum_{j=1}^k \sigma_{ij} S_t^{(i)} dW_t^{(j)}, \quad \text{for } i=1,\ldots,k, \quad (1)$$

where $\{\mu_i : i=1,\ldots,k\}$ and $\{\sigma_{ij} : i,j=1,\ldots,k\}$ are constants. The price of the risk-free asset is assumed to evolve according to

$$dB_t = r B_t\, dt, \quad (2)$$

where $r\geq 0$. We assume that $\mu_i > r$ for all $i=1,\ldots,k$.

An investment policy is a (column) vector control process $f=\{f_t : t\geq 0\}$ in R^k with individual components $f_t^{(i)}$, $i=1,\ldots,k$, where $f_t^{(i)}$ is the fraction (we use f for fraction) or proportion of the investor's wealth invested in the risky stock i at time t, for $i=1,\ldots,k$, with the remainder invested in the risk-free bond. It is assumed that $\{f_t, t\geq 0\}$ is a suitable, admissible $\mathcal{F}_t$-adapted control process, i.e., f_t is a nonanticipative function that satisfies $\int_0^T f_t' f_t\, dt < \infty$ a.s. for every $T<\infty$. We place no other restrictions on f, for example, we allow $\sum_{i=1}^k f_t^{(i)} \geq 1$, whereby the investor is leveraged and has borrowed to purchase the stocks, as well as $f_t^{(i)} < 0$, whereby the investor is selling stock i short.

Let X_t^f denote the *wealth* of the investor at time t under policy f, with $X_0 = x$. Since any amount not invested in the risky stock is held in the bond, this process then evolves as

$$\begin{aligned} dX_t^f &= X_t^f \left(\sum_{i=1}^k f_t^{(i)} \frac{dS_t^{(i)}}{S_t^{(i)}}\right) + X_t^f \left(1 - \sum_{i=1}^k f_t^{(i)}\right) \frac{dB_t}{B_t} \\ &= X_t^f \left(r + \sum_{i=1}^k f_t^{(i)} (\mu_i - r)\right) dt \\ &\quad + X_t^f \sum_{i=1}^k \sum_{j=1}^k f_t^{(i)} \sigma_{ij}\, dW_t^{(j)} \end{aligned} \quad (3)$$

upon substituting from (1) and (2). This is the wealth equation first studied by Merton (1971).

If we introduce now the matrix $\sigma = (\sigma)_{ij}$ and the column vectors $\mu = (\mu_1,\ldots,\mu_k)'$, $\mathbf{1} = (1,\ldots,1)'$, and $W_t = (W_t^{(1)},\ldots,W_t^{(k)})'$, we can rewrite the wealth process of (3) as

$$dX_t^f = X_t^f[(r + f_t'(\mu - r\mathbf{1}))\, dt + f_t' \sigma\, dW_t]. \quad (4)$$

For the sequel, we will also need the matrix $\Sigma = \sigma\sigma'$. It is assumed for the sequel that the square matrix σ is of full rank, hence σ^{-1} (and Σ^{-1}) exists.

2.1. The Benchmark Portfolio

As described above, our interest lies in determining investment strategies that are optimal relative to the performance of a benchmark. The benchmark we work with here is the wealth associated with another portfolio strategy $\pi = (\pi(1),\ldots,\pi(k))'$ where $\pi(i)$ denotes the fraction of the benchmark wealth invested in the i-th stock. Accordingly, the benchmark portfolio evolves similarly to (4), as

$$dX_t^\pi = X_t^\pi[(r + \pi'(\mu - r\mathbf{1}))\, dt + \pi' \sigma\, dW_t]. \quad (5)$$

For example, if $\pi(i)=0$ for each $i=1,\ldots,k$, then the benchmark is simply "cash," which is the relevant benchmark in a variety of situations (see Litterman and Winkelman 1996). Alternatively, if $\pi(i)=1$, with $\pi(j)=0$ for all $j\neq i$, then the benchmark is just the i-th stock or asset. We note that while the problems studied in this paper can in fact be treated with more general benchmarks (and in more general settings), here we only consider the constant coefficients case for analytical and economic simplicity.

2.2. Optimal Growth

Let π^* be the constant vector defined by

$$\pi^* = \Sigma^{-1}(\mu - r\mathbf{1}). \tag{6}$$

The vector π^* plays a fundamental role in the theory of finance (see Merton 1990, Ch. 6) and will also play a fundamental role in the sequel. Following Merton (1990), we refer to the vector π^* as the *optimal-growth* portfolio strategy. The reason for this is the policy $f_t = \pi^*$, for all t, has many optimality properties associated with it in an ordinary portfolio setting (where there is no benchmark) that are relevant for growth-related objectives. In particular, for an investor whose wealth evolves according to (4), and who is not concerned with performance relative to any benchmark, (i) π^* maximizes the expected logarithm of terminal wealth, for any fixed terminal time T, hence (ii) π^* maximizes the (actual and expected) rate at which wealth compounds. More interesting, perhaps, and certainly more relevant to our concerns here is the property (iii): π^* *minimizes the expected time* until any given level of wealth is achieved (needless to say, so long as that level is greater than the initial wealth). Merton (1990, Ch. 6) contains a comprehensive review of these properties (see also Browne 1998 for further optimality properties). Given these results, it is not surprising that the policy π^* has extended optimality properties in our benchmark-based model as well, as we show below. Most relevant to our concerns is the fact that indeed π^* is the policy that minimizes the expected time until the benchmark portfolio strategy is beaten by any given percentage (see Corollary 1 below). The fact that this holds for *any* benchmark, however, severely limits the applicability of this result in that it does not provide any insight for the active portfolio manager as to the role the benchmark plays in the investment decision, and as such might not be a reasonable objective for an active portfolio manager.

We note also that the ratio of the wealth process of any (admissible) portfolio strategy to the wealth process determined by the optimal growth strategy is a supermartingale. This fact has important consequences for pricing contingent claims, as described, for example, in Merton (1990, Ch. 6).

2.3. Active Portfolio Management

There are of course many possible objectives related to outperforming a benchmark. Here, as in Browne (1999a), we consider objectives related solely to the achievement of relative performance goals and shortfalls or drawdowns. Specifically, for numbers l, u with $lX_0^\pi < X_0^f < uX_0^\pi$, we say that performance goal u is reached (relative to the benchmark asset allocation strategy π) if $X_t^f = uX_t^\pi$, for some $t > 0$, and that performance shortfall level l occurs if $X_t^f = lX_t^\pi$ for some $t > 0$. The active portfolio management problems considered for an *incomplete market* (i.e., where there are more sources of risk than there are traded securities) in Browne (1999a) are: (i) maximizing the probability that performance goal u is reached before shortfall l occurs; (ii) minimizing the expected time until the performance goal u is reached; (iii) maximizing the expected time until shortfall l is reached; (iv) maximizing the expected discounted reward obtained upon achieving goal u; and (v) minimizing the expected discounted penalty paid upon falling to shortfall level l. Among other scenarios, these objectives are relevant to institutional money managers, whose performance is typically judged by the return on their managed portfolio relative to the return of a benchmark. Browne (1999a) showed that the optimal strategy for each of these objectives was in fact a *constant proportions* asset allocation strategy. While constant proportion strategies are optimal in a variety of other settings as well, many professional investors object to them on the grounds that they do not take into account the wealth level of the investor.

In this paper we consider an extended fractional objective that relates the probability in Objective (i) to the expected time in Objective (ii). It turns out that for this objective the optimal strategy is no longer constant, but rather state dependent. In particular, the policy is hyperbolic in the state variable, where the relevant state variable is the *ratio of the wealth process to the benchmark.* We use standard techniques of stochastic control theory (e.g., Krylov 1980) to establish our results since this ratio is a controlled diffusion.

In particular, because X_t^f is a controlled geometric Brownian motion, and X_t^π is another geometric Brownian motion, it follows directly that the ratio process, Z^f, where $Z_t^f = X_t^f / X_t^\pi$, is also a controlled geometric Brownian motion. Specifically, a direct application to Ito's formula gives:

PROPOSITION 1. *For X_t^f, X_t^π defined by* (4) *and* (5), *let Z_t^f be defined by $Z_t^f = X_t^f / X_t^\pi$. Then, using the definition of the vector π^* as given in* (6), *we have*

$$dZ_t^f = Z_t^f (f_t - \pi)' \Sigma (\pi^* - \pi)\, dt + Z_t^f (f_t - \pi)' \sigma\, dW_t. \quad (7)$$

Alternatively, in integral form we have

$$Z_t^f = Z_0 \exp\left\{ \int_0^t (f_s - \pi)' \Sigma \left(\pi^* - \tfrac{1}{2}(f_s + \pi)\right) ds + \int_0^t (f_s - \pi)' \Sigma (f_s - \pi)\, dW_s \right\}. \quad (8)$$

Next, we provide a general theorem in stochastic optimal control for the process $\{Z_t^f, t \geq 0\}$ of (7) that covers the specific problems treated here as special cases.

2.4. Optimal Control

The active portfolio management problems considered in this paper are special cases of optimal control problems of the following (Dirichlet-type) form: For the ratio process $\{Z_t^f, t \geq 0\}$ given by (7), let

$$\tau_x^f = \inf\{t > 0 : Z_t^f = x\} \quad (9)$$

denote the first hitting time to the point x under a specific policy $f = \{f_t, t \geq 0\}$. For given numbers l, u, with $l < Z_0 < u$, let $\tau^f = \min\{\tau_l^f, \tau_u^f\}$ denote the first escape time from the interval (l, u), under this policy f.

For a given real bounded continuous function $g(z)$ and a function $h(z)$ given for $z = l, z = u$, with $h(u) < \infty$, let $v^f(z)$ be the reward function under policy f, defined by

$$v^f(z) = E_z \left(\int_0^{\tau^f} g(Z_t^f)\, dt + h(Z_{\tau^f}^f) \right), \quad (10)$$

with

$$v(z) = \sup_{f \in \mathcal{G}} v^f(z), \quad \text{and} \quad f_v^*(z) = \arg\sup_{f \in \mathcal{G}} v^f(z) \quad (11)$$

denoting, respectively, the optimal value function and associated optimal control function, where $\mathcal{G}$ denotes the set of admissible controls. (Here and in the sequel, we use the notations $P_z(\cdot)$ and $E_z(\cdot)$ as shorthand for $P(\cdot | Z_0 = z)$ and $E(\cdot | Z_0 = z)$.) We note at the outset that we are only interested in controls (and initial values z) for which $v^f(z) < \infty$.

REMARK 1. Observe that the reward functional in (10) is sufficiently general to cover a variety of goal-related objectives. For example, the probability of beating the benchmark before being beaten by it, following a given strategy $\{f_t\}$, i.e., $P_z(\tau_u^f < \tau_l^f)$, is a special case with $g(\cdot) = 0$, $h(u) = 1$ and $h(l) = 0$. Similarly, by taking $g(\cdot) = 1$ and $h(u) = 0 = h(l)$, we obtain $E_z(\tau^f)$. Related optimal control problems have been treated previously in various forms for a variety of models. In particular see Pestien and Sudderth (1985), Heath et al. (1987), and Browne (1995, 1997, 1999a). Related stochastic differential games are treated in Browne (2000).

As a matter of notation, we note first that here, and throughout the remainder of the paper, the parameter γ will be defined by

$$\gamma = \gamma(\pi) = (\pi^* - \pi)' \Sigma (\pi^* - \pi)/2, \quad (12)$$

where π^* is the optimal-growth policy of (6) and π is the benchmark under consideration.

The following theorem, which is a special case of the more general Theorem 1 in Browne (1999a), shows that the optimal value function is the solution to a particular nonlinear ordinary differential equation with Dirichlet boundary conditions, and that the optimal policy is given in terms of the first two derivatives of this solution.

THEOREM 1. *Suppose that $w(z)$ is twice continuously differentiable with the first two derivatives given by w_z and w_{zz}, and is the strictly concave increasing* (*i.e.*, $w_z > 0$ *and* $w_{zz} < 0$) *solution to the nonlinear Dirichlet problem*

$$-\gamma \frac{w_z^2(z)}{w_{zz}(z)} + g(z) = 0, \quad \text{for } l < z < u, \quad (13)$$

with

$$w(l) = h(l), \quad \text{and} \quad w(u) = h(u), \quad (14)$$

and satisfies the following three conditions:

(i) $(w_z^2(z)/w_{zz}(z))$ *is bounded for all z in* (l, u);

(ii) for every $t \geq 0$, and every admissible policy f, we have

$$E\int_0^t (Z_s^f w_z(Z_s^f))^2\, ds < \infty; \text{ and} \tag{15}$$

(iii) $(w_z(z)/w_{zz}(z))$ is locally Lipschitz-continuous. Then $w(z)$ is the optimal value function, i.e., $w(z) = v(z)$, and, moreover, the optimal control vector, f_v^*, can then be written as

$$f_v^*(z) = \pi - (\pi^* - \pi)\left(\frac{w_z(z)}{z w_{zz}(z)}\right), \tag{16}$$

where π^* is the vector defined in (6).

The utility of Theorem 1 for our purposes is that for various choices of the functions $g(\cdot)$ and $h(\cdot)$, it addresses the objective problems discussed earlier. Moreover, it shows that for each of these problems, all we need do is solve the ordinary differential Equation (13) and then take the appropriate derivatives to determine the optimal control by (16). Conditions (i), (ii), and (iii) are just technical conditions that ensure integrability of certain functionals, which in turn ensure optimality. We will not discuss them further here, but the interested reader should see Browne (1999a). Conditions (i) and (iii) are easy to check, and while Condition (ii) seems potentially hard to verify, for the cases considered here it is in fact easy, as demonstrated below.

Remark 2. Observe that the representation of the optimal control vector $f_v^*(z)$ of (16) demonstrates that the optimal portfolio strategy consists of two distinct parts: (i) the *tracking* component π and (ii) the *active* component, $-(\pi^* - \pi) w_z/(z w_{zz})$. Because w is increasing and concave in z, the active component associated with asset i is positive if $\pi^*(i) > \pi(i)$, and negative if $\pi^*(i) < \pi(i)$. That is, the active manager will invest more heavily in asset i than the benchmark if the benchmark is underinvested in asset i relative to the vector π^*, and vice versa. The extent to which this occurs depends on the specifics of the value function $w(z)$.

Remark 3. As noted earlier, Theorem 1 is a special case of a more general result in Browne 1999a, and as such we do not provide a formal proof. However, to provide some insight into the result, observe that the Hamilton–Jacobi–Bellman (HJB) optimality equation of dynamic programming for maximizing $v^f(z)$ of (10) over control policies f, to be solved for an optimal value function v, is

$$\sup_f \{(f-\pi)'\,\Sigma(\pi^*-\pi)\, z v_z + (f-\pi)'\,\Sigma(f-\pi)\, z^2 v_{zz} + g\} = 0, \tag{17}$$

subject to the Dirichlet boundary conditions $v(l) = h(l)$ and $v(u) = h(u)$ (see, e.g., Krylov 1980, Theorem 1.4.5, or Fleming and Soner 1993, §IV.5).

Assuming now that (17) admits a classical solution with $v_z > 0$ and $v_{zz} < 0$, we may then use standard calculus to optimize with respect to f in (17) to obtain the optimal control function $f_v^*(x)$ of (16), with $v = w$. When (16) is then substituted back into (17) and simplified, we obtain the nonlinear Dirichlet problem of (13) (with $v = w$). To complete the proof now, one only needs to verify that the solution to the HJB equation is indeed the optimal value function, and hence that the policy f_v^* is indeed optimal. A verification argument based on the martingale optimality principle given in Browne (1999a) covers the case at hand, provided that Conditions (i), (ii), and (iii) hold. A stochastic differential game-theoretic version of the verification argument and martingale principle appears in Browne (2000).

3. Minimizing the Expected Time to Beat the Benchmark

3.1. Optimality of π^*

We can use Theorem 1 to show that, as claimed earlier, the ordinary optimal growth portfolio policy, π^* of (6), is indeed also optimal for minimizing the expected time to beat the benchmark by any predetermined amount, *regardless of the underlying benchmark strategy* π. Indeed, we state this formally in the following corollary to Theorem 1.

Corollary 1. *Let $G^*(z) = \inf_f E_z(\tau_u^f)$ with optimizer $f^*(z) = \arg\inf_f E_z(\tau_u^f)$. Then for any $\pi \neq \pi^*$, and γ as defined in* (12), *we have*

$$G^*(z) = \frac{1}{\gamma}\ln\left(\frac{u}{z}\right), \quad \text{with } f^*(z) = \pi^*, \text{ for all } z \leq u. \tag{18}$$

Proof. Observe first that while Theorem 1 is stated in terms of a maximization problem, it obviously contains the minimization case, as we can apply

Theorem 1 to $\tilde{G}(z) = \sup_f\{-E_z(\tau_u^f)\}$, and then recognize that $G^* = -\tilde{G}$. As such, Theorem 1 applied with $g(z) = 1$ and $h(u) = 0$ shows that G^* must solve the ordinary differential equation

$$-\gamma\frac{G_z^2(z)}{G_{zz}(z)} + 1 = 0, \tag{19}$$

together with the boundary condition $G^*(u) = 0$. Moreover, G^* must be *convex decreasing* (since it is the solution to a minimization problem). It is easy to substitute the claimed values from (18) into (19) to verify that in fact that is the case. Furthermore, we have $G_z^*/zG_{zz}^* = -1$, and as such (16) of Theorem 1 shows that the optimal control for this case reduces to π^*.

It remains to verify whether the Conditions (i) (ii) and (iii) of Theorem 1 hold: It is clear that (i) and (iii) hold. Condition (ii) is seen to hold for this case since we have $dG^*(z)/dz = -1/(z\gamma)$, and as such Requirement (15) reduces here to $\int_0^t \gamma^{-2} ds < \infty$, which holds trivially. □

Remark 4. We have just shown that the ordinary optimal growth policy, π^*, minimizes the expected time to a goal in the presence of a benchmark. To see that this same policy maximizes logarithmic utility of the ratio for the investor, simply observe that $\sup_f\{E[\ln(Z_T^f)]\} = \sup_f\{E[\ln(X_T^f)]\} - E[\ln(X_T^\pi)]$.

3.2. Properties of the Growth-Optimal Ratio

When π^* is substituted back into (7), we obtain the following stochastic differential equation for the ratio of the growth-optimal wealth to the benchmark

$$dZ_t(\pi^*, \pi) = Z_t(\pi^*, \pi)[\gamma dt + (\pi^* - \pi)'\sigma\, dW_t], \tag{20}$$

which implies that under π^*, this ratio process is the geometric Brownian motion given by

$$Z_t(\pi^*, \pi) = Z_0 \exp\{\gamma t + (\pi^* - \pi)'\sigma W_t\}. \tag{21}$$

While we did not consider a lower shortfall barrier, l, in the development above proving optimality of the standard growth-optimal portfolio policy, it is of importance in many applications to consider one, since many investors indeed are interested in avoiding substantial shortfalls. In the following proposition we give two fundamental results for the wealth process for an investor following the optimal-growth strategy π^*: (i) the *probability* that the investment goal u is reached before a shortfall of size l occurs; and (ii) the expected time to escape the interval (l, u) (which is not the same as the expected time to the goal).

Proposition 2. *For the process $Z_t(\pi^*, \pi)$ of (21), let τ denote the first escape time from the interval (l, u), and let $\theta(z : l, u)$ denote the probability of ''successful'' escape, i.e., $\tau = \inf\{t : Z_t(\pi^*, \pi) \notin (l, u)\}$, and $\theta(z : l, u) = P_z(Z_\tau(\pi^*, \pi) = u)$. Then $\theta(z)$ is given by*

$$\theta(z : l, u) = \frac{u}{z}\left(\frac{z - l}{u - l}\right). \tag{22}$$

Also, the expected time of first escape from the interval is given by

$$E_z(\tau(\pi^*, \pi)) = \gamma^{-1}\left[\theta(z : l, u)\ln\left(\frac{u}{l}\right) - \ln\left(\frac{z}{l}\right)\right]. \tag{23}$$

Thus, we see that while π^* is the policy under which any given investment goal will be reached in minimal expected time, and hence in a sense maximizes expected return, it does so with a risk of a shortfall of size l occurring with probability $1 - \theta(z : l, u)$, where $\theta(z : l, u)$ is the probability given in (22).

Remark 5. It is important to note that the probability $\theta(z : l, u)$ is *independent of any of the underlying parameters* associated with the underlying model. In particular, this probability is independent of the *benchmark* policy π. (For example, the probability of the ratio doubling $(u = 2z)$ before being halved $(l = 0.5z)$ is always $\frac{2}{3}$.) This limits the usefulness of the optimal-growth policy in the active portfolio management setting, since it provides no guide to the manager in how to choose a benchmark. Of course, the expected time to escape the interval (l, u), given by (23), does depend on the underlying parameters, but only through the parameter γ.

Remark 6. Observe that for $l = 0$, we obtain $\theta(z : 0, u) = 1$, and the expected escape time from the interval $E_z(\tau(\pi^*, \pi))$ reduces, as it should, to the optimal expected first passage time to the upper barrier $E_z(\tau_u(\pi^*, \pi))$. Of course, for $l > 0$, the expected hitting time of the optimal ratio process to the lower shortfall level, $E_z(\tau_l(\pi^*, \pi))$, is infinite due to the fact that the drift in (20) is positive.

REMARK 7. The probability in (22) and the expected hitting time in (23) can be established directly via a variety of different ways. Most directly we have the following lemma for geometric Brownian motion:

LEMMA 1. *Let X_t denote a geometric Brownian motion that satisfies*

$$dX_t = mX_t dt + \sqrt{2s}\, X_t dW_t, \quad \text{with } X_0 = x \tag{24}$$

and let $\tau_z = \inf\{t > 0 : X_t = z\}$. For $0 \le a \le x \le b$, define now $\tau = \min\{\tau_a, \tau_b\}$, and

$$K(x) = P_x(\tau_b < \tau_a) \equiv P_x(\tau = \tau_b) \tag{25}$$

$$H(x) = E_x(\tau). \tag{26}$$

Then for $m \ne s$ we have

$$K(x) = \frac{x^v - a^v}{b^v - a^v}, \quad \text{where } v = 1 - \frac{m}{s}, \tag{27}$$

$$H(x) = \frac{1}{m - s}\left(K(x)\ln\left(\frac{b}{a}\right) - \ln\left(\frac{x}{a}\right)\right), \tag{28}$$

while for $m = s$ we have

$$K(x) = \frac{\ln(x/a)}{\ln(b/a)}, \tag{29}$$

$$H(x) = \frac{1}{2m}\left([(\ln b)^2 - (\ln a)^2]\frac{\ln(x/a)}{\ln(b/a)} - [(\ln x)^2 - (\ln a)^2]\right). \tag{30}$$

PROOF. Recognize first that $\{X_t\}$ is a diffusion process with drift function $\mu(x) = mx$ and diffusion function $\sigma^2(x) = 2sx$. Therefore, if follows from elementary results about one-dimensional diffusions that K and H are the unique solutions to the respective (Dirichlet) problems:

$$mxK_x + sx^2K_{xx} = 0: \quad K(a) = 0,\ K(b) = 1 \tag{31}$$

$$mxH_x + sx^2H_{xx} + 1 = 0: \quad H(a) = 0,\ H(b) = 0 \tag{32}$$

(see e.g., Karlin and Taylor 1981, pp. 192–193). The general solution to the second order differential equation in the left-hand side of (31), for $m \ne s$, is $K(x) = C_1x^v + C_2$, and the boundary conditions determine the constants C_1, C_2, as in (27). For $m = s$, the general solution is $C_1 \ln x + C_2$, and the boundary conditions give (29).

The general solution to the left-hand side of (32) is $C_1 + C_2x^v - (m - s)^{-1}\ln x$ for $m \ne s$, and from the boundary conditions we determine C_1 and C_2, giving

$$\frac{1}{m - s}\left(\frac{1}{b^v - a^v}\right)[-(b^v - a^v)\ln x + (x^v - a^v)\ln b + (b^v - x^v)\ln a],$$

which is equivalent to (28) above when we simplify using the definition of $K(x)$. For $m = s$, the general solution of (32) is $C_1 - (\frac{1}{2}m)(\ln x)^2 + C_2 \ln x$, which gives (30) after applying the boundary conditions. □

Using this lemma for our purposes, we first note that (20) is distributionally equivalent to a diffusion that evolves according to the stochastic differential equation

$$dX_t = X_t(2\gamma dt + \sqrt{2\gamma}\, dW_t),$$

where W is an independent one-dimensional Brownian motion. As such, identify $m = 2\gamma$, $s = \gamma$, and hence $v = -1$, and then using $b = u$ and $a = l$, (22) and (23) follow upon substitution. □

4. Risk-Related Objectives

4.1. Objective Function and Optimal Policy

The results of the previous section indicate that the optimal growth strategy, π^*, regardless of its many optimality properties in the ordinary portfolio setting, may not be appropriate for an active portfolio manager who cares about downside risk as well as upside growth relative to a benchmark.

As such, in this section we treat an objective that is perhaps more appropriate in that it allows the active portfolio manager to incorporate the shortfall probability directly in the risk/return tradeoff relative to expected growth. In particular, we consider a linear tradeoff between the shortfall probability and the expected time to get to the surplus level. More specifically, we now treat the following objective: For given nonnegative constants α and β, let

$$V^*(z) = \sup_f \{\alpha P_z(Z^f_{\tau^f} = u) - \beta E_z(\tau^f)\}. \quad (33)$$

As we will show, the optimal dynamic portfolio strategy for this objective is no longer a constant asset allocation strategy. Rather, as we will establish below, the optimal strategy hyperbolically modulates the fractions invested in the risky assets by the level of the ratio process.

THEOREM 2. *Let $V^*(z)$ denote the optimal value function in* (33), *and let $f^*_V(z)$ denote the associated optimal control function. Then, $V^*(z)$ is given by*

$$V^*(z) = \alpha \ln\left(\frac{z+b}{l+b}\right) \Big/ \ln\left(\frac{u+b}{l+b}\right), \quad \text{for } l \le z \le u, \quad (34)$$

where the scalar $b = b(\alpha, \beta : u, l, \gamma)$ is given by

$$b = \frac{ue^{-\gamma\alpha/\beta} - l}{1 - e^{-\gamma\alpha/\beta}}. \quad (35)$$

The optimal portfolio policy, $f^*_V(z)$, is given by

$$f^*_V(z) = \pi - (\pi^* - \pi)\left(1 + \frac{b}{z}\right) \equiv \pi^* + (\pi^* - \pi)\frac{b}{z}, \quad (36)$$

where b is given by (35), *π^* is the ordinary optimal growth vector given earlier in* (6),

$$\pi^* = \Sigma^{-1}(\mu - r\mathbf{1}),$$

and π is the benchmark strategy.

REMARK 8. Observe that the portfolio strategy $f^*_V(z)$ is a state-dependent policy that is inversely modulated by the level of the ratio process Z. The final representation in (36) shows that the policy is composed of two parts: First it just uses the optimal-growth policy π^*, and then multiplies the difference between the optimal-growth policy π^* and the tracking portfolio π by the correction term b/z. The sign of the correction factor is determined by the sign of b. Some direct manipulations on (35) shows that the sign of b is the sign of

$$\frac{1}{\gamma}\ln\left(\frac{u}{l}\right) - \frac{\alpha}{\beta}.$$

Comparison now with (18) reveals that we can write this quantity as $G^*(l) - \alpha/\beta$, where $G^*(l)$ is the minimal possible expected time to get from the shortfall level l to the surplus goal u. Thus, b is positive (negative) if the ratio α/β is less (more) than this minimal expected time.

Observe further that if $b > 0$, then the active manager invests more heavily in the i-th stock than does π^* so long as the benchmark is underinvested in that stock relative to the optimal-growth policy, i.e., so long as $\pi^*(i) > \pi(i)$.

Finally, note that (35) shows that we must always have $b \ge -l$.

PROOF. Theorem 1 applies directly to this case with $g(z) = -\beta$, with $h(u) = \alpha$ and $h(l) = 0$. As such, we require that V^* be the concave increasing solution to the nonlinear Dirichlet problem:

$$-\gamma\frac{V_z^2}{V_{zz}} - \beta = 0 \quad \text{for} \quad l < z < u, \quad (37)$$

and that V^* satisfy the boundary conditions $V^*(u) = \alpha$ and $V^*(l) = 0$.

The general form of the solution to the nonlinear ordinary differential equation in (37) is of the form $V(z) = (\beta/\gamma)\ln(z + C_1) + C_2$, where C_1 and C_2 are arbitrary constants which we can choose to match the boundary conditions. Observe that this function is concave. The boundary condition $V^*(u) = \alpha$ determines that $C_2 = \alpha - (\beta/g)\ln(u + C_1)$, and the boundary condition $V^*(l) = 0$ determines that $C_1 = b$, where b is the constant in (35). As such, the value function is given explicitly by

$$V^*(z) = \alpha + \frac{\beta}{\gamma}\ln\left[(z+b)\frac{(1 - e^{-\gamma\alpha/\beta})}{u - l}\right]. \quad (38)$$

Observe now that we can invert the relation in (35) to write $\beta/\gamma = \alpha/\ln[(u+b)/(l+b)]$, which when placed into (38) and then simplified, gives the value function in the form that it is given in (34).

For this value function, the conditions given in Theorem 1: (i) and (iii) are seen to hold directly, and (ii) can be established by first noting that $(zw_z)^2 = (\beta/\gamma)^2(z/[z+b])^2$, and then by using the fact that since $z/[u+b] \le z/[z+b] \le z/[l+b]$, we have

$$E\int_0^t (Z_s^f w_z(Z_s^f))^2 ds = \left(\frac{\beta}{\gamma}\right)^2 E\int_0^t \left(\frac{Z_s^f}{Z_s^f+b}\right)^2 ds$$

$$< \left(\frac{\beta/\gamma}{l+b}\right)^2 \int_0^t E(Z_s^f)^2\, ds < \infty,$$

where the final inequality holds by the assumption of admissibility.

Because we now have an explicit form for the optimal policy, we may place it into (16) to obtain the optimal control vector given by $f_V^*(z)$ of (36). □

4.2. The Optimal Process

Observe that when we place the optimal control $f_V^*(Z_t)$ of (36) back into (7), we obtain an optimal-wealth process $Z(f_V^*, \pi)$ which we will denote by Z^*, that follows the stochastic differential equation, using the definition of γ from (12),

$$dZ_t^* = 2\gamma(Z_t^*+b)\,dt + (Z_t^*+b)(\pi^*-\pi)'\sigma dW_t. \quad (39)$$

The unique strong solution to (39) is given by

$$Z_t^* = (Z_0+b)\exp\{\gamma t + (\pi^*-\pi)'\,\sigma W_t\} - b. \quad (40)$$

Comparison with the results on the optimal-growth policy of the last section shows that we can write this in terms of the optimal-growth ratio as

$$Z_t^* = \left(1+\frac{b}{Z_0}\right)Z_t(\pi^*, \pi) - b, \quad (41)$$

where $Z_t(\pi^*, \pi)$ is the ratio of the ordinary optimal-growth wealth to the benchmark, as given in (21).

In the next proposition we list the following two properties of the optimal ratio process Z^*: (i) the probability of reaching the upper surplus goal u before the lower shortfall level l; and (ii) the expected time it takes to escape from the interval (l, u). Recall that the scalar b depends on the benchmark through the parameter γ, and that we always have $b \ge -l$.

PROPOSITION 3. *Let $\{Z_t^*,\ t\ge 0\}$ denote the optimal-wealth process associated with the control function $f_V^*(z)$ of* (36), *and let τ^* denote the associated first escape time from the interval (l, u), i.e., $\tau^* = \inf\{t : Z_t^* \notin (l, u)\}$. Also, let $\phi(z : b, l, u)$ denote the probability of successful escape from the interval, i.e., $\phi(z : b, l, u) = P_z(Z_{\tau^*}^* = u)$. Then*

$$\phi(z : b, l, u) = \frac{(z-l)(u+b)}{(z+b)(u-l)}. \quad (42)$$

The expected time of the first escape is given by

$$E_z(\tau^*) = \frac{1}{\gamma}\left[\phi(z : b, l, u)\ln\left(\frac{u+b}{l+b}\right) - \ln\left(\frac{z+b}{l+b}\right)\right]. \quad (43)$$

REMARK 9. Observe that the probability in (42) is now dependent on the benchmark policy π through the parameter b. Note further that as intuition would suggest, the probability ϕ in (42) is larger than the associated probability for the optimal-growth strategy, θ obtained earlier in (22), if $b<0$, and smaller if $b>0$.

PROOF. The results above can be established directly from the earlier Lemma 1 applied to the process Z_t^*+b, because as (41) exhibits, Z_t^*+b is a geometric Brownian motion, with initial state Z_0+b. In fact, (41) implies that Z_t^*+b is distributionally equivalent to a multiple of the optimal-growth ratio described in the last section, i.e., $Z_t^*+b \stackrel{d}{=} (1+b/Z_0)Z_t(\pi^*, \pi)$. As such, we have

$$\phi(z : b, l, u) \equiv P_z(Z_{\tau^*}^* = u) = P_z(Z_{\tau^*}^*+b = u+b)$$

$$= P_z\left(\left(1+\frac{b}{z}\right)Z_\tau(\pi^*, \pi) = u+b\right) \quad (44)$$

where in the latter τ denotes the first escape time of the process $(1+b/z)Z_t(\pi^*, \pi)$ from the interval $(l+b, u+b)$. As such, the results of the previous section allows us to evaluate this latter probability in terms of the function $\theta(\cdot)$ defined earlier in (22). Specifically, the argument in (44) shows that

$$\phi(z : b, l, u) = \theta\left(z : \frac{l+b}{1+b/z}, \frac{u+b}{1+b/z}\right)$$

$$\equiv \theta(z+b : l+b, u+b), \quad (45)$$

and indeed (42) is obtained when we substitute appropriately into (22).

The optimal expected hitting time, $E_z(\tau^*)$ of (43) is derived directly from the fact that under the optimal policy f_V^*, the value function is

$$V^*(z) = \alpha P_z(Z_{\tau^*}^* = u) - \beta E_z(\tau^*),$$

and therefore

$$E_z(\tau^*) = \frac{1}{\beta}[\alpha P_z(Z^*_{\tau^*} = u) - V^*(z)]. \tag{46}$$

Substituting now for $P_z(Z^*_{\tau^*} = u)$ from (42) and for $V^*(z)$ from (34), and using $1/\beta = \ln[(u+b)/(l+b)]/(\alpha\gamma)$ gives (43). □

4.3. Risk-Constrained Minimal Time

The results of the previous section can now be applied directly to the active portfolio management case where the shortfall probability is prespecified. Specifically, suppose that the shortfall probability is prespecified to the active manager to be no more than $1-p$, where p is a given number between 0 and 1, i.e., the active manager is told that he must have $P_z(Z^f_{\tau^f} = l) \le 1 - p$, or equivalently, that he must have $P_z(Z^f_{\tau^f} = u) \ge p$. The risk-constrained active portfolio management problem is now to minimize the expected time to beat the benchmark subject to a constraint on the shortfall probability, specifically, to find the strategy $\{f^*_t, t \ge 0\}$ that minimizes $E(\tau^f)$ subject to $P(Z^f_{\tau^f} = u) \ge p$, where p is a given number in (0, 1). This is now related to the gambling problem first solved in Gottlieb (1985). Following Gottlieb, we observe first that the dual of the risk-constrained active portfolio management problem is to maximize the probability that $P(Z^f_{\tau^f} = u)$ subject to a constraint on $E(\tau^f)$. Moreover, observe that should a solution exist, then the constraint will be met at equality, and so we would have $P(Z^f_{\tau^f} = u) = p$.

Let us write the solution to the dual problem, should it exist, as

$$\Psi(z) = \sup_f [P_z(Z^f_{\tau^f} = u) - \beta E_z \tau^f], \tag{47}$$

where β is now the value of a Lagrangian multiplier.

The control problem in (47) is a special case of the problem treated above with $\alpha = 1$, and as such, from (34) we know that the solution is given by

$$\Psi(z) = \Psi(z) = \ln\left(\frac{z+\tilde{b}}{l+\tilde{b}}\right) \Big/ \ln\left(\frac{u+\tilde{b}}{l+\tilde{b}}\right), \tag{48}$$

where $\tilde{b}$ is the value of b in (35) evaluated at $\alpha = 1$, i.e., $\tilde{b} = b(1, \beta)$. The value of β, the Lagrangian multiplier, will be determined from the risk constraint $P(Z^f_{\tau^f} = u) = p$.

We also know from (36) that the associated risk-constrained optimal portfolio strategy is given by

$$f^*(z) = \pi^* + (\pi^* - \pi)\frac{\tilde{b}}{z}. \tag{49}$$

Observe now that we may invert the identity for b in (35) in this case to write the unknown β in terms of the unknown $\tilde{b}$ as

$$\beta = \gamma\left[\ln\left(\frac{u+\tilde{b}}{l+\tilde{b}}\right)\right]^{-1}. \tag{50}$$

Because we require $\beta > 0$, this implies that we require $\tilde{b} \ge -l$ (we always need $b \ge -l$, to ensure that the probability $\phi(z : \cdot, \cdot)$ does not exceed 1, i.e., to keep $\phi \le 1$).

To determine the value of $\tilde{b}$, we can use the risk constraint evaluated at the initial time 0, i.e., *set*

$$\phi(Z_0; \tilde{b}) = p.$$

Using (42) with $b = \tilde{b}$, this gives us

$$\frac{(Z_0 - l)(u + \tilde{b})}{(Z_0 + \tilde{b})(u - l)} = p,$$

which in turn now can be solved for $\tilde{b}$, giving

$$\tilde{b} = \tilde{b}(p, Z_0, u, l) = \frac{pZ_0(u-l) - u(Z_0 - l)}{Z_0 - l - p(u-l)}. \tag{51}$$

Remark 9. Observe that

$$\inf_{\tilde{b} > -l} \phi(Z_0 : \tilde{b}) = \frac{Z_0 - l}{u - l}$$

and as such, the risk-constrained problem is feasible only for an initial probability level p that satisfies

$$p > \frac{Z_0 - l}{u - l}.$$

Observe too that for $p = 1$, $\tilde{b}$ reduces to $\tilde{b} = -l$, which makes the lower barrier l unattainable as in many ''portfolio insurance'' models (see Browne 1997). The insurance level $\tilde{b}$ in (51) is positive for values of p satisfying

$$\frac{Z_0 - l}{u - l} < p < \frac{u}{Z_0}\left(\frac{Z_0 - l}{u - l}\right)$$

and $\tilde{b}$ is negative for larger values in the region

$$p > \frac{u}{Z_0}\left(\frac{Z_0 - l}{u - l}\right) \equiv \theta(Z_0),$$

where $\theta(Z_0)$ is the initial probability that the optimal growth wealth/ratio hits u before l, as given in (22). Thus, as intuition suggests, to have a higher ''success'' probability than the optimal-growth strategy, the active portfolio manager must take less risk and invest less (since $\tilde{b} < 0$) than the ordinary optimal-growth investor.

The optimal expected hitting time, $E_z(\tau^*)$ can be obtained directly as

$$E_z(\tau^*) = \frac{1}{\gamma}\left[\frac{z-l}{z+\tilde{b}}\left(\frac{u+\tilde{b}}{u-l}\right)\ln\left(\frac{u+\tilde{b}}{l+\tilde{b}}\right) - \ln\left(\frac{z+\tilde{b}}{l+\tilde{b}}\right)\right]. \tag{52}$$

5. Conclusions

We have studied a goal-related objective for the problem of outperforming a benchmark. The objective relates the time to outperform the benchmark to the shortfall probability. The resulting optimal policy is state-dependent in an intuitive way not captured by previous studies where the optimal policy was of a constant proportions type. Moreover, this optimal policy directly addresses and alleviates one of the undesirable features, in the benchmark outperformance problem, of the ordinary optimal-growth policy, whose shortfall probability was shown to be independent of the benchmark and any other model-specific parameter. As a special case of this objective, we studied the problem of minimizing the expected first passage time subject to a constrained probability of successful escape.

References

Black, F., A. F. Perold. 1992. Theory of constant proportion portfolio insurance. *J. Econom. Dynam. and Control* **16** 403–426.

Browne, S. 1995. Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin. *Math. Oper. Res.* **20** 937–958.

——. 1997. Survival and growth with a fixed liability: Optimal portfolios in continuous time. *Math. Oper. Res.* **22** 468–493.

——. 1998. The return on investment from proportional portfolio strategies. *Adv. Appl. Probab.* **30**(1) 216–238.

——. 1999a. Beating a moving target: Optimal portfolio strategies for outperforming a stochastic benchmark. *Finance and Stochastics* **3** 275–294.

——. 1999b. Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Adv. Appl. Probab.* **31** 551–577.

——. 1999c. The risk and reward of minimizing shortfall probability. *J. Portfolio Management* **25**(4) 76–85.

——. 2000 Stochastic differential portfolio games. *J. Appl. Probab.* In press, **37**(1).

Duffie, D. 1996. *Dynamic Asset Pricing Theory*, 2nd ed. Princeton University Press, Princeton, NJ.

Fleming, W. H., H. M. Soner. 1993. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York.

Gottlieb, G. 1985. An optimal betting strategy for repeated games. *J. Appl. Probab.* **22** 787–795.

Grinold, R. C., R. N. Kahn. 1995. *Active Portfolio Management.* Irwin, Probus, IL.

Heath, D., S. Orey, V. Pestien, W. Sudderth. 1987. Minimizing or maximizing the expected time to reach zero. *SIAM J. Control and Optim.* **25**(1) 195–205.

Karlin, S., H. M. Taylor. 1981. *A Second Course in Stochastic Processes*. Academic, New York.

Krylov, N. V. 1980. *Controlled Diffusion Processes.* Springer-Verlag, New York.

Litterman, R., K. Winkelmann. 1996. Managing market exposure. *J. Portfolio Management* **22**(4) 32–48.

Merton, R. 1971. Optimum consumption and portfolio rules in a continuous time model. *J. Econom. Theory* **3** 373–413.

——. 1990. *Continuous Time Finance*. Blackwell, MA.

Perold, A. F., W. F. Sharpe. 1988. Dynamic strategies for asset allocation. *Financial Anal. J.* **44**(1) 16–27.

Pestien, V. C., W. D. Sudderth. 1985. Continuous-time red and black: How to control a diffusion to a goal. *Math. Oper. Res.* **10**(4) 599–611.

Sharpe, W. F., G. F. Alexander, J. V. Bailey. 1995. *Investments*, 5th ed. Prentice Hall, NJ.

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