Opportunities Knocking: Residual Income Valuation of an Adaptive Firm

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Maintaining a competitive edge requires a firm to replace deteriorating business lines with new projects. Accordingly, part of a firm's value resides in its ability to exploit new opportunities. This paper incorporates adaptation into Ohlson's residual income valuation framework and obtains an adaptation-adjusted valuation formula. Although parsimoniously cast, the model makes two predictions that are consistent with phenomena reported in the empirical literature: earnings convexity and complementarity. Moreover, the Appendix introduces an Equivalence Theorem relating Modigliani-Miller dividend invariance, complementarity, and convexity.

1. Overview

1.1 Motivation

Existing securities valuation models, notably the class of models introduced by Ohlson (1995) and Feltham and Ohlson (1995), make assumptions leading to strictly *linear* relationships between price and accounting numbers. These valuation models do not capture many of the phenomena described in the burgeoning empirical literature documenting nonlinear relations between price or returns and accounting numbers (e.g., Barth, Beaver, and Landsman [1998]; Berger, Ofek, and Swary [1996]; Burgstahler and Dichev [1997]; Easton [1999]; Hayn [1995]; Kinney, Burgstahler, and Martin [1999]; Subramanyam and Wild [1996]).

A combination of many factors is likely responsible for such nonlinearities. In fact, any driver that affects reported earnings and book value potentially impacts the relationship between price and accounting numbers. Features of the financial reporting system, including earnings management and conservatism, undoubtedly

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influence the relationship. Economic factors also impact this relationship by virtue of their influence on prospective cash flows and how they are reflected in accounting numbers. How the firm responds to its macroeconomic environment may affect its earnings persistence and drive deviations from Ohlson's "LIM" dynamic.

This paper focuses on one particular microeconomic driver: adaptation. This is not to suggest that accounting or other economic factors are less significant. On the contrary, the lasting value of the model presented here is probably as a prototype for inspiring the innovation of more comprehensive models.

The themes championed here are straightforward. First, abnormal earnings contingency—such as that imposed by adaptation—generates a nonlinear relationship between price and accounting numbers. Second, the nonlinear relationship induced by adaptation is schematically consistent with the earnings convexity and complementarity previously identified with adaptation by Beaver, Barth, Landsman and others in the empirical literature. Third and most notably, the adaptation solution exemplifies a broader equivalence theorem (Theorem 4 in the Appendix) relating Modigliani-Miller dividend invariance, earnings convexity, and complementarity.

This paper calls attention to two generally underappreciated dimensions of residual income valuation. First, it highlights the *contingent* nature of prospective residual earnings.[1] While Ohlson's general framework invites contingency, the popularity of his linear LIM examples have inspired a widespread fixation on linear models. While the incentives for empiricists to focus on regression-ready linear models are certainly understandable, linear modeling can never truly *explain* the observed pattern of relations between price and accounting numbers.

Contingency permeates equity. It is a driving force underlying limited liability, bankruptcy, corporate reorganizations, executive compensation, and investing and financing policy. For example, the simplest imaginable dynamic incorporating Chapter 7 style (contractual) bankruptcy is contingent (Yee [1999]):

$$x_{t+1}^a = \begin{cases} \omega x_t^a + \nu_{t+1} & \text{if } fa_t \geq 0; \\ 0 \text{ permanently} & \text{if } fa_t < 0; \end{cases}$$

where x_t^a is abnormal earnings and fa_t the firm's free cash.

The second dimension this paper highlights is methodology. This paper demonstrates that dynamic programming, so fruitfully exploited by economists to develop intertemporal consumption-investment models of economic behavior (Merton [1990]; Stokey and Lucas [1989]), can also be employed for accounting-based securities valuation. In particular, solution techniques employed in the existing residual income valuation literature would be completely stumped by contingency. However, as illustrated below, the Bellman equation is the key to evaluating the Edwards-Bell-Ohlson (EBO) formula when abnormal earnings are contingent.

1. Feltham (1996) also examines contingency but in a different dimension: the cleanliness of the clean surplus relation. As defined by Feltham, contingent cleanliness occurs when the NPV of employee stock options are recognized immediately at the contract date as opposed to, for instance, the exercise date.

While the methodology to be presented employs the same analytic tools (dynamic programming) as contingent claims pricing in finance, it must be kept in mind that the model developed herein is a *securities*—not a derivative securities—valuation model. Traditional contingent claims valuation assumes an exogenously priced underlying asset, and does nothing more than relate the value of the derivative claim(s) to the prices of the underlying assets. In contrast, the securities valuation formula herein relates value directly to accounting information. The exogenous variables here are not asset prices, but the state prices defining the risk neutral measure, the clean surplus accounting variables (dividends, book value, and earnings), and the parameters characterizing the quality of switching opportunities and costs.

Here, the key foundational difference between traditional contingent claims and residual income valuation stems from the pricing of the firm's opportunity set. Both contingent claims and residual income valuation strive for grounding on the hard sediment of the no-arbitrage principle (Feltham and Ohlson [1999]). On positing a riskless numeraire security, both benefit from the existence of an equivalent risk neutral measure, which is tantamount to a list of primitive state prices. Contingent claims prices a derivative security by valuing a self-financing replicating portfolio created from underlying assets priced by market equilibrium. The valuation of the adaptive firm proceeds analogously subject to one caveat: if the underlying assets—the firm's prospective projects—are not publicly traded, their prices (or equivalently the associated state prices) are exogenously imposed rather than endogenously determined by market equilibrium. In this event, the value derived for the adaptive firm is a "real" rather than bona fide option.[2] Notwithstanding this caveat, which is material, residual income valuation as implemented herein shares all remaining foundational elements with traditional contingent claims valuation based on the no-arbitrage principle.

1.2 Adaptation

Continuing operations undoubtedly comprise a significant portion of firm value. Capital budgeting options, which permeate the activities of all firms, also contribute to firm value (e.g., Luehrman [1998]). That is, the prerogative to terminate and replace existing projects with more lucrative ones has value. The source of this value resides in expected future cash flows from *prospective* investments to be judiciously drawn from a firm's investment opportunity set at the appropriate time.

A firm's market niche—grounded on the firm's recognized and intangible assets such as human capital, intellectual property, reputation, and output capacity—facilitates the quality of its investment opportunity set. Accordingly, the quality of opportunities imminently accessible to each firm is idiosyncratic. The task under-

2. I thank Suresh Govindaraj and Mike Harrison for discussions on this issue.

taken in this paper is to model the impact on the accounting-based valuation function of a firm's prerogative to optimally exploit its opportunity set.

This paper refers to a perpetual ability to terminate existing projects and replace them with new ones from its opportunity set as a firm's *adaptation option*, and a firm that exercises such options as being *adaptive*. In effect, the adaptive firm holds a perpetual stream of project-switching options, which it exercises optimally. This terminology was used in the financial accounting literature by Burgstahler and Dichev (1997) and adopted by Wysocki (1998). Burgstahler and Dichev's characterization of adaptation subsumes the *abandonment* concept of Barth, Beaver, and Landsman (1998); Berger, Ofek, and Swary (1996); Hayn (1995); and Subramanyam and Wild (1996); the latter groups of authors focus primarily on earnings-poor or financially distressed firms, where "abandonment for liquidation value" is the predominant form of adaptation. To be sure, most of these authors refer to the liquidation of specific *projects* rather than of the whole firm. However, in keeping with parsimony, the stylized firm modeled in this paper is permitted to operate only one project at any time so no distinction will be made between a project and a firm.

While adaptation (as the term is used here) encompasses the abandonment of an existing project, it also includes reinvestment in a new project. Internet firms come to mind as timely anecdotal illustrations of financially *non*distressed firms with significant adaptation value. The (negligible or negative) ROEs from the ongoing operations of a Yahoo!, Commerce One, or eBAY do not justify their market capitalizations. Rather, their market valuations are supported, presumably, by their claim as early Internet niche occupiers favorably positioned take advantage of unrevealed future opportunities that may arise as the Internet develops. If so, then the main driver of Internet market capitalizations is adaptability.

1.3 Earnings Convexity and Complementarity

Earnings convexity and complementarity are formally defined in Appendix A.1. As empirically identified by Burgstahler and Dichev (1997) and others, earnings convexity is the convex relation between price and reported earnings. Burgstahler and Dichev hypothesize that adaptation is responsible for convexity. Their story is intuitive: since unsatisfactory-earnings firms are more likely to switch projects, small or negative earnings are much less persistent than large positive earnings. Hence price, which anticipates *future* earnings, is less sensitive to small or negative contemporaneous earnings, which have no intertemporal forecasting power.

Focusing on a sample of financially distressed firms, Barth, Beaver and Landsman (1998) (hereafter BBL) report striking evidence of what I shall refer to as *complementarity*. Complementarity is the contrary (or complementary) behavior of the valuation coefficients of reported book value and earnings. As BBL found, the valuation coefficient and incremental explanatory power of book value increases while those of earnings simultaneously decrease as financial distress increases.

They attribute complementarity to the proposition that the balance sheet reflects liquidation value (the value available to pay off creditors), while income reflects not only recognized assets but also earnings generated by off-balance-sheet intangibles, which may not be convertible into liquidation value. The BBL story is one of liquidation and abandonment. From the perspective that liquidation and abandonment is a special case of adaptation where existing projects are replaced by a project with normal earnings, it is also an adaptation story.

Earnings convexity and complementarity are *not* independent phenomena. In fact, whenever Modigliani-Miller dividend invariance holds, earnings convexity and complementarity tend to occur together. As explained in Appendix A.1, they are related by the Equivalence Theorem:

> In any Markovian accounting system MM dividend irrelevance implies that *earnings convexity is equivalent to complementarity*. That is, under these assumptions convexity and complementarity must occur together or not at all.

This means that in a Markovian MM world Burgstahler and Dichev's earnings convexity and BBL's complementarity are two different guises of the same phenomena.

Hints of the Equivalence Theorem are readily evident even in Ohlson's original model (1995), which is manifestly MM invariant. Assuming the linear dynamic $x^a_{t+1} = \omega x^a_t + \varepsilon_{t+1}$, Ohlson finds that $V = b_t + [\kappa/(R - 1)]x^a_t$, where $\kappa \equiv (R - 1)\omega/(R - \omega)$. After transforming to the earnings x_t coordinate system, V is written as

$$\hat{V} = \underbrace{\beta}_{1-\kappa} b_t + \underbrace{\xi}_{\kappa\phi} x_t + \underbrace{\delta}_{-\kappa} d_t,$$

where $\phi \equiv R/(R - 1)$. The caret over $\hat{V}_t$ indicates it is a function of the x_t rather than the x^a_t coordinate system. It is easy to see that the pricing multiples of $\hat{V}$ obey two sum rules:

$$\phi \times \beta + \xi = \phi. \tag{1}$$

$$\phi \times \delta + \xi = 0. \tag{2}$$

Surprisingly, these sum rules prevail well beyond Ohlson's special solution. Indeed, it turns out that they are universal features of MM invariant residual income valuation. As shown in Theorem 2 in the Appendix, they govern *any* Modigliani-Miller invariant residual income valuation model, linear or nonlinear, subject to mild technical constraints on the accounting system. In fact, while these sum rules break down when Modigliani-Miller invariance is violated, the degree of deviation from these sum rules can be *quantitatively* related to how and by how much MM is violated in an analytically rigorous way (Yee [1999]).

The first sum rule, eq. (1), leads to the Equivalence Theorem as follows. Earnings convexity means that the earnings coefficient ξ increases with increasing earnings. Maintaining eq. (1) requires the book value and earnings coefficients, β and ξ, to behave in contrary ways: if ξ increases β must commensurately decrease to preserve the sum rule. Hence, earnings convexity implies the book value multiple, β, decreases with increasing earnings. Therefore, earnings convexity and complementarity occur together due to eq. (1). Since eq. (1) is a direct consequence of MM invariance (subject to mild conditions on the accounting system), the simultaneous occurrence of convexity and complementarity is implied by MM invariance.

A truly adequate appreciation of the Equivalence Theorem requires a more elaborate formulation of Modigliani-Miller invariance in a multiperiod accounting framework than I am able to present here. Nonetheless, I hope the introduction to the Equivalence Theorem given here and in the appendix is enough to shed light on why the adaptation solution presented herein exhibits both convexity and complementarity. The point to take away is: Convexity and complementarity occur together in the adaptation solution, *not* due to any novel feature of adaptation per se, but because the adaptation solution is MM invariant. Yee (1999) offers a more comprehensive formulation of accounting-based Modigliani-Miller invariance and the Equivalence Theorem than is possible here.

1.4 Modeling Assumptions

In keeping with Ohlson's (1995) spirit of parsimony, strong assumptions are adopted not only to facilitate a clean closed-form analytical solution, but also to avoid obscuring adaptation as the mechanism responsible for driving nonlinearity in this model. The primitive assumptions of the model are:

- *No dividends arbitrage*: this implies the dividend discount formula

$$V = \sum_{\tau=1}^{\infty} R^{-\tau} E(d_{t+\tau}), \tag{3}$$

  where R is the risk-free discounting factor, presumed constant, and E is the equivalent risk-neutral expectation operator;
- *Clean surplus relation*: $b_{t+1} = b_t + x_{t+1} - d_{t+1}$, where b is book value, x earnings, and d dividends;
- *Efficient accounting* (Ohlson [1999]): in the risk neutral measure, earnings is constructed so that abnormal earnings of a continuing project persist with the dynamic $x^a_{t+1} = \omega x^a_t$;
- *Rationality:* whenever it is advantageous to do so, the firm replaces its existing project with a project randomly drawn from its investment opportunity set (to be characterized later);
- *Stationarity:* the quality distribution of projects in the firm's investment opportunity set does not change with time;
- *Financing constraint:* the firm operates one and only one project at any given time independent of its accrued book value.

In light of these strong assumptions, the model should be interpreted schematically rather than literally.

Following Feltham and Ohlson (1999), applying the clean surplus relation to eq. (3) yields the EBO residual income valuation formula in the equivalent risk neutral representation:

$$V(b_t, x_t^a) = b_t + \sum_{\tau=1}^{\infty} R^{-\tau} E(x_{t+\tau}^a). \tag{4}$$

Implicit in the derivation of eq. (4) is the transversality condition

$$\lim_{T \to \infty} R^{-T} E(b_T) = 0,$$

which is effectively a modest lower bound on the dividend payout rate.

In eq. (4) abnormal earnings x_t^a is the earnings above $(R - 1)b_{t-1}$, the firm's risk-free cost of capital. According to the no arbitrage principle, the risk neutral expectation operator, E, is the correct expectation to use in the discounted dividends formula. (Complete markets are not assumed, so E is not necessarily unique outside the state space span of the arbitrageable assets.) Its risk adjusted counterpart, E^{real}, is the "econometric" expectation operator[3] which automatically reflects investors' risk preferences. Recovering an E expectation from an E^{real} expectation requires subtracting out the risk premia component of market returns. This means that for accounting systems of interest typically

$$E^{real}(x_{t+\tau}^a) > E(x_{t+\tau}^a) \quad \forall\, \tau > 0.$$

Throughout this paper, I will be working in the risk neutral measure. In particular, the abnormal earnings dynamic will always be stated in the risk neutral coordinate system. This has important implications because relative to the "real" measure the economically identical process would have an altogether different appearance. For example, suppose the abnormal earnings dynamic in the risk neutral measure is $x_{t+1}^a = \omega x_t^a + \tilde{\varepsilon}_{t+1}$, where $E(\tilde{\varepsilon}_{t+1}) = 0$. This dynamic corresponds to a conservative accounting system in the real measure (Ohlson [1999]). That is,

$$E^{real}(\tilde{\varepsilon}_{t+\tau}) > E(\tilde{\varepsilon}_{t+\tau}) = 0 \quad \forall\, \tau > 0,$$

so, relative to the real measure, abnormal earnings do not revert to zero even though they do in the equivalent risk neutral measure.

Finally, as in Ohlson (1995) and Feltham and Ohlson (1995), my firm is an all-equity firm with no leverage. Equivalently, one can allow nontrivial capital structure, but interpret $V(b_t, x_t^a)$ as total firm value (the value of equity plus debt); b_t as the book value of assets (not shareholders' equity); and x_t as earnings before interest.

Section 2 reformulates $V(b_t, x_t^a)$ as the solution to a dynamic programming problem. Specifically, Section 2.1 poses a simple contingent residual earnings dynamic consistent with the aforelisted assumptions. Section 2.2 describes the Bellman equa-

3. Equally weighted cross-sectional averages of real-world data estimate E^{real}, not E. Taking an E average requires backing out investors' risk preferences.

tion for the firm's investment decision problem. Finally, Section 2.3 describes the analytical formula for $V(b_t,x_t^a)$ obtained by solving the Bellman equation. The remainder of the article discusses properties of the solution which formally bridge the chasm between Ohlson's residual income valuation framework and the empirically observed nonlinear effects of adaptation.

1.5 Synopsis of Results

As derived in Section 2, $V(b_t, x_t^a)$ is of the functional form[4]

$$V(b_t, x_t^a) = b_t + \alpha\,[\gamma\,(x_t^a) + \lambda(x_t^a)x_t^a], \tag{5}$$

where b_t is the book value reported at date t; x_t^a is the reported abnormal earnings; and $\alpha \equiv \omega/(R - \omega)$, Ohlson's pricing multiple for abnormal earnings (Ohlson [1995]). Coefficients $\gamma(x_t^a)$ and $\lambda(x_t^a)$, whose formulas are derived in Section 2 (Lemma 1), are near-constants, which vary only mildly with x_t^a when $x_t^a >> 0$. Closer to $x_t^a \sim 0$, these coefficients are more sensitive to x_t^a. It is precisely λ's gradual rise with x_t^a that causes $V(b_t,x_t^a)$ to be convex. Coefficients γ and λ (and hence V) are also sensitive to earnings persistence parameter ω, discounting factor R, project switching cost κ (to be defined), and the quality distribution F of adaptation opportunities (to be defined).

As depicted in Figure 1, adaptability has its greatest impact on $V(b_t,x_t^a)$ when $x_t^a \leq 0$ (with b_t held fixed). Adaptability gives a firm the ability to discontinue an unprofitable project and switch to, on expectation, one of average profitability. Hence negative abnormal earnings are not persistent. Accordingly, $V(b',x_t^a)$ is insensitive to x_t^a when it is small. Indeed Figure 1 confirms that $V(b',x_t^a)$ is flat when x_t^a is small. Algebraically, Section 2 finds that $\lambda(0) = 0$ while $\alpha\gamma(0)$, which is tantamount to the adaptive firm's minimum value, obeys

$$\alpha\gamma(0) = V(b_t, 0) - b_t > 0.$$

Independent of its current profitability, the firm is not worth less than $\alpha\gamma(0)$ by virtue of its adaptation opportunities, which comes with the ability to get rid of net loss projects; $\alpha\gamma(0)$ reflects the value of a firm's assets—recognized as well as intangible—assuming perpetual adaptation.

Section 2 finds that in the extreme of high abnormal earnings $\gamma(x_t^a) \mapsto 0$ and $\lambda(x_t^a) \mapsto 1$. This implies that $V(b_t,x_t^a)$ converges asymptotically to the linear Ohlson formula:

$$\lim_{x_t^a \mapsto \infty} V(b_t, x_t^a) \sim b_t + \alpha x_t^a.$$

This is not surprising. When persistent residual earnings are very large, the firm will not find it favorable to switch projects for a long time. Accordingly, the firm

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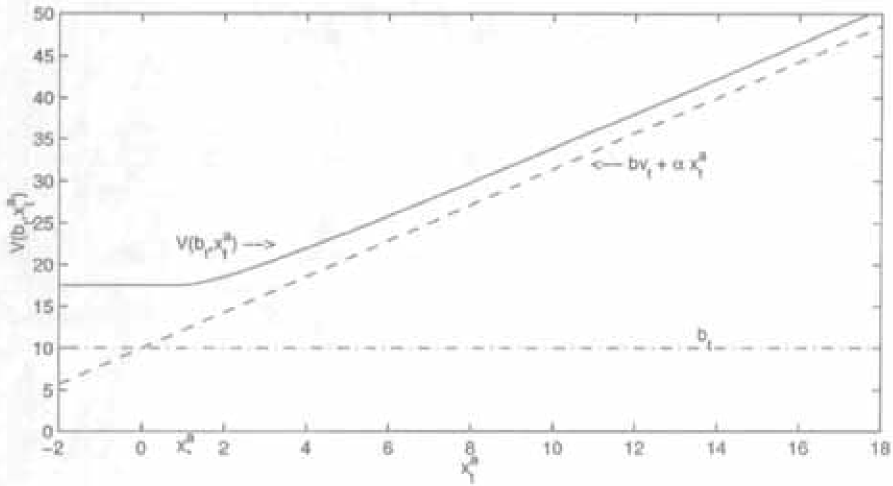
4. By Theorem 1 in Appendix A.1, this is a special case of the most general form consistent with MM invariance and a Markovian abnormal earnings dynamic.

FIGURE 1

Residual Earnings Dependence of Adaptive Firm Value



As will be shown in Section 2, the value $V(b_t,x_t^a)$ of an adaptive firm is a continuous, increasing convex function of its abnormal earnings x_t^a (and hence also its earnings x_t). For comparison, the sloped dotted line represents the Ohlson value $b_t + \alpha x_t^a$ of a nonadaptive firm with the same (b_t,x_t^a). The horizontal dotted line indicates the size of book value b_t. Notably, $\lim_{x_t^a \to -\infty} V(b_t, x_t^a - b_t >> 0$. This difference is due to the impact of adaptability, which is especially large for an earnings-poor firm, for which abandonment has its greatest value. Note that b_t is held fixed in the plot.

is valued as if it were a nonadaptive firm irrevocably committed to its ongoing operations.

In the intermediate regime between small and infinitely large residual earnings, the coefficients take on their intermediate values:

$$0 < \gamma(x_t^a) \leq \gamma(0), \quad 0 < \lambda(x_t^a) < 1.$$

In this regime, adaptation endows a firm with a perpetual stream of put options to replace its residual income stream (minus any switching cost) with a substitute drawn from the firm's opportunity set. An indicator of the impact of adaptability is the difference between $V(b_t,x_t^a)$ and the value $b_t + \alpha x_t^a$ of an Ohlson firm. As depicted in Figure 6 in Section 2.4, this indicator is convex and schematically reminiscent of a (superposition of) put option(s).

To facilitate comparison with empirical studies, it is necessary to recast eq. (5) as a relation between book value, reported earnings, and firm value. Here, application of the clean surplus relation reveals that eq. (5) is algebraically equivalent to[5]

5. While their relation to γ and λ does not specifically concern us here, for the record:

$$d_t + \hat{V}(b_t, d_t, x_t) = A + B \times (b + d)_t + C \times x_t \qquad (6)$$

Valuation coefficients A, B, and C are *not* constants: in addition to varying mildly with $(b + d)_t$ and x_t, they are sensitive to earnings persistence, the discount rate, switching costs, and the quality distribution of switching opportunities. The caret over $\hat{V}(b_t, x_t)$ distinguishes[6] the functional dependence of $\hat{V}$ on x_t from the functional dependence of V on x_t^a.

It is no accident that the (RHS) of Eq. (6) only depends on book value through the combination $(b + d)_t$. Corollary 1 in Appendix A.1 says that in an MM world with Markovian abnormal earnings the cum-dividend valuation function $d_t + \hat{V}$ can depend on book value *only* via the cum-dividend combination $(b + d)_t$.

When $x_t \mapsto \infty$ with b_t held fixed, abnormal earnings are infinitely large. In this limit, the firm will not find it advantageous to switch projects in the foreseeable future and the impact of adaptation must be negligible. Hence, $\hat{V}(b_t, x_t)$ must asymptotically recover its linear Ohlson counterpart in the $x_t \mapsto \infty$ limit:

$$\lim_{x_t \mapsto \infty} d_t + \hat{V}(b_t, x_t) \sim \frac{R}{R - \omega}[(1 - \omega)(b + d)_t + \omega x_t]. \qquad (7)$$

Indeed, as depicted in Figure 2, valuation coefficients A, B, and C become insensitive to earnings and converge to their respective Ohlson values in the large x_t limit.

In contrast, all three coefficients are sensitive to variation in earnings x_t when the firm is adaptive.[7] Figure 2 reveals two notable features which deserve mention. First, as x_t falls below some critical value, C drops rather dramatically to zero. Simultaneously, B rises (but less dramatically). Hence, this model successfully postdicts the book value and earnings complementarity phenomenon reported by BBL.[8]

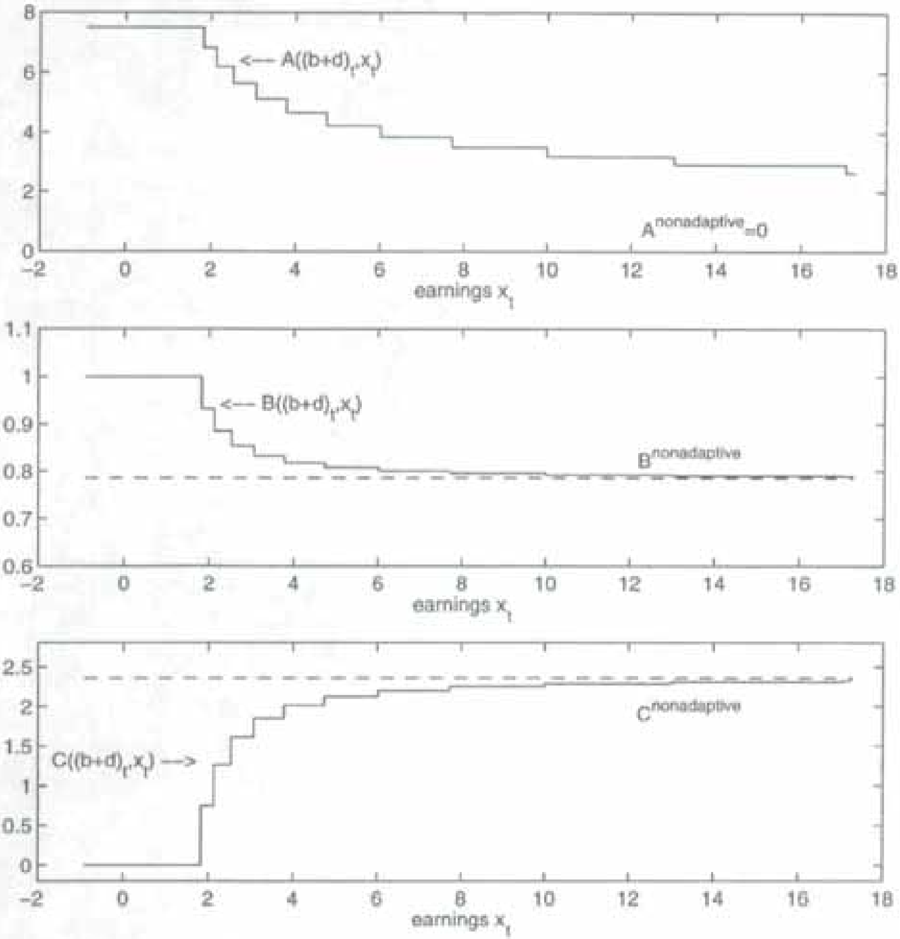
$$A((b + d)_t, x_t) = \alpha\, \gamma\, (Rx_t - (R - 1)(b + d)_t),$$
$$B((b + d)_t, x_t) = 1 - \alpha\, (R - 1)\, \lambda\, (Rx_t - (R - 1)(b + d)_t),$$
$$C((b + d)_t, x_t) = \alpha\, R\, \lambda\, (Rx_t - (R - 1)(b + d)_t).$$

6. That is, while $\hat{V}(b_t, x_t) = V(b_t, x_t^a)$ in value when $x_t^a = Rx_t - (R - 1)(b + d)_t$, the way $\hat{V}(b_t, x_t)$ varies with x_t is not the same as how $V(b_t, x_t^a)$ varies with x_t^a.

7. The stepwise (rather than smooth) variation of A, B, and C in Figure 2 is an artifact of one of my modeling assumptions. As described in Section 2, for reasons of modeling parsimony I will permit an adaptive firm to switch projects only once per financial reporting period (in fact, only during the reporting date). As a result, temporal aggregation in the firm's decision process results in a stepwise rather than smooth valuation function. Since real-world firms can switch projects any time (and more than once) during financial reporting periods, the stepwise nature of A, B, and C are not expected to be reflected in empirical measurements.

8. The Ohlson limit, eq. (7), may be interpreted as being consistent with complementarity if smaller values of ω are associated with earnings-poor firms, which are more likely to switch projects. Then eq. (7) "predicts" that the earnings coefficient is smaller for earnings-poor firms while the book value coefficient is bigger. However, from a theoretical perspective this story is not satisfactory because the variation of ω with x_t is introduced in an ad hoc (and qualitative) way. In this article, adaptation frequency will be endogenously determined.

FIGURE 2



The solid lines display the earnings dependence of eq. (6) valuation coefficients *A*, *B*, and *C*; the horizontal dotted lines indicate their respective values in the Ohlson (nonadaptive-firm) limits, where these coefficients are x_t-invariant. Consistent with complementarity, when x_t drops below some critical value (approximate $x_t = 4$ in the Figure), earnings coefficient *C* falls rather dramatically to zero. Simultaneously, book value coefficient *B* rises (but less dramatically). Because these Figures are drawn assuming a $\nu = 0$ information dynamic, that *A* is nonzero indicates that, even in the absence of exogenous information variables, adaptability confers $V(b_t, x_t^a)$ a firm-specific *A* term, which mimics the contribution of a ν.

1.6 Dormant Value

The second feature is that $V(0,0) = \alpha\gamma(0)$, call it the adaptive firm's dormant value, is nonzero as previously noted. Dormant value reflects the value added by the mere possibility of taking advantage of new opportunities. In the adaptation model, dormant value depends on firm-specific features like earnings persistence, switching costs, and the quality distribution of project opportunities.

Unused Internet domain names (and other forms of legal property rights) are notable contemporary examples of assets with dormant value but, perhaps, no recognized book value or near-term earnings potential. As described by Yee (1998), attractive "dot com" domain addresses are a natural resource in limited supply—not unlike prestigious real estate locations. While there may be many retailers selling, for instance, toys over the Internet, only one merchant can do it through each of the Internet addresses "toys.com" or "etoys.com." The merchants who do business through these two generic addresses have obvious marketing advantages over competitors: potential e-shoppers will tend to type in these two easy-to-remember names as first resorts. Moreover, following the lead of successful law and brokerage firms who stake out prestigious Manhattan addresses, online merchants can signal their superior quality or resources by staking out the preferred Internet domains. With this in mind, it is no surprise that a merchant has recently paid as much as 7.5 million dollars for one attractive Internet address, and that another owner "has already received a bid for higher than $7.5 million for America.com. The seller rejected it" (*Media Grok* [1999]).

2. Valuing an Adaptive Firm: The Model

This section constructs the adaptation model in its most parsimonious guise: a one-project firm with a Markovian abnormal earnings dynamic which is contingent only on[9] the contemporaneous abnormal earnings value x_t^a, and a stream of inexhaustible project opportunities beckoning adoption. In this setting, the adaptation-adjusted LIM and its associated clean surplus relation are derived in Section 2.1.

As constructed, evolution of the adaptation-adjusted LIM depends on the adaptive firm's strategy. Accordingly, Section 2.2 assumes the firm adapts by optimally switching projects to maximize the EBO expression

$$b_t + \sum_{\tau=1}^{\infty} R^{-\tau} E(x_{t+\tau}^a | x_t^a).$$

Solution of this intertemporal optimization problem in a Bellman equation approach determines the value, $V(b_t, x_t^a)$, of the EBO expression evaluated at the residual

9. This means that accounting is efficient, there is no dividend signaling, and the exogenous information variable v_t is omitted.

income generated by the optimal adaptation strategy. Accordingly, Section 2.3 interprets $V(b_t, x_t^a)$ as the value of an optimally adaptive firm.

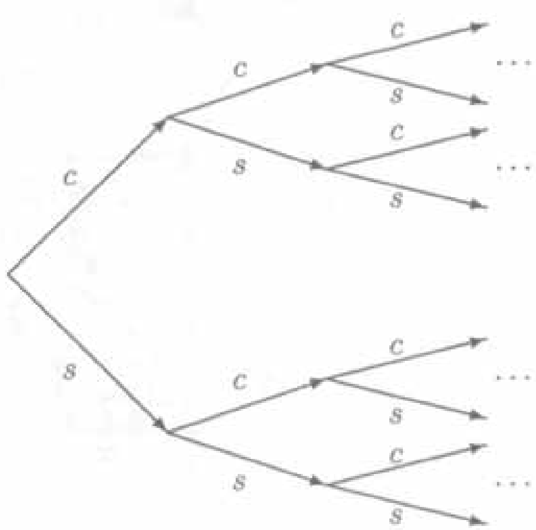
Features ascribed to $V(b_t, x_t^a)$ in Section 1 are formally verified in Sections 2.3–2.6. Moreover, comparative statics analysis of the formula obtained for (b_t, x_t^a) reveals adaptation-adjusted relations between features of firm value and abnormal earnings persistence, the discounting factor, switching costs, and the quality distribution of project opportunities available to the firm.

2.1 Adaptation Operationalized

As indicated in the decision tree of Figure 3, an adaptive firm faces a new decision at each reporting date t. At each t, it must take one of two actions: continue (c) or switch (s). If the firm switches, it abandons its existing project in favor of a new one with (hopefully) improved x_{t+1}^a. If the firm continues, it sticks with its existing project for (at least) one additional period. Switching or continuing, however, does not impose any obligation beyond the next period: a firm that continues at t is free to switch or not at $t + 1$ (and vice versa). As a direct consequence of its idiosyncratic sequence of decisions, each adaptive firm evolves along one of the possible branches of the perpetual decision tree depicted in Figure 3.

This characterization of adaptability captures the inevitability that all rational firms eventually must switch projects as residual earnings from ongoing projects

FIGURE 3



An adaptive firm faces an infinite horizon decision tree. As posited, at every financial reporting date t the firm must choose whether to continue (c) with its existing project or to switch (s) to a new project. As a result of its sequence of choices, the firm evolves along a particular path of the tree.

dissipate and fall below acceptable levels. In particular, an optimally adaptive firm will switch projects when the present expected value of switching exceeds the present expected value of continuing. As a result, the value of an adaptive firm incorporates not just the discounted expected value of its ongoing project (what Ohlson's formula captures), but also the value of the firm's omnipresent privilege to abandon that project and switch to a more promising one whenever it is favorable to do so.

The present value of an adaptive firm depends on which particular investment path in the Figure 3 tree the firm ultimately chooses to evolve along. However, because the firm makes its choice at each date τ based on the best information available to it at that time, the firm (and capital markets) cannot predict based on the available information at $t < \tau$ what its choice will be at τ. As a result, the value $V(b_t, x_t^a)$ of the firm at t is an expected value averaged over all of its prospective evolutionary paths. In contrast, the Ohlson LIM valuation model obligates a firm to continue with its existing project. Accordingly, the Ohlson firm is irrevocably committed to a $c \to c \to c \to \ldots$ strategy ad infinitem. As it is, the value-added by the prerogative to pick a different, more lucrative path in the tree is what distinguishes $V(b_t, x_t^a)$ from Ohlson's linear $b_t + \alpha x_t^a$ formula.

Figure 4 summarizes the adaptive firm's two choices at date t. If the firm continues with its existing project, then $x_{t+1}^a = \omega x_t^a$ and, by clean surplus, $b_{t+1}^c = Rb_t + \omega x_t^a - d_{t+1}$. If the firm switches, the firm must (1) abandon (irreversibly) its existing project, (2) pay a one-time switching cost κ, and (3) reinvest its capital in a new untried project yielding randomly drawn abnormal earnings $\tilde{y}$ (on top of κ). The new abnormal earnings $\tilde{y}$ will not be revealed until $t + 1$, when the firm faces its next decision.

Thus the adaptive firm's residual earnings dynamic is

$$x_{t+1}^a = \begin{cases} x_{t+1}^{ac} \equiv \omega x_t^a & \text{if continue} \\ x_{t+1}^{as} \equiv \tilde{y} & \text{if switch.} \end{cases} \tag{8}$$

Interpreting this dynamic requires recalling that x_t^a is the value of abnormal earnings in the equivalent risk-neutral measure. In this coordinate system, prospective earnings are automatically risk-adjusted so that in a competitive economy one expects earnings to revert to the risk-free cost of capital. Since x_t^a is, by construction, earnings minus the risk-free accounting cost of capital, one expects abnormal earnings x_t^a to revert to zero in the long run—as implied by eq. (8). Thus, eq. (8) is economically justified provided that real accounting earnings are unbiased after adjusting for risk.[10] Ohlson calls such an idealized accounting system "efficient accounting" (Ohlson 1999).

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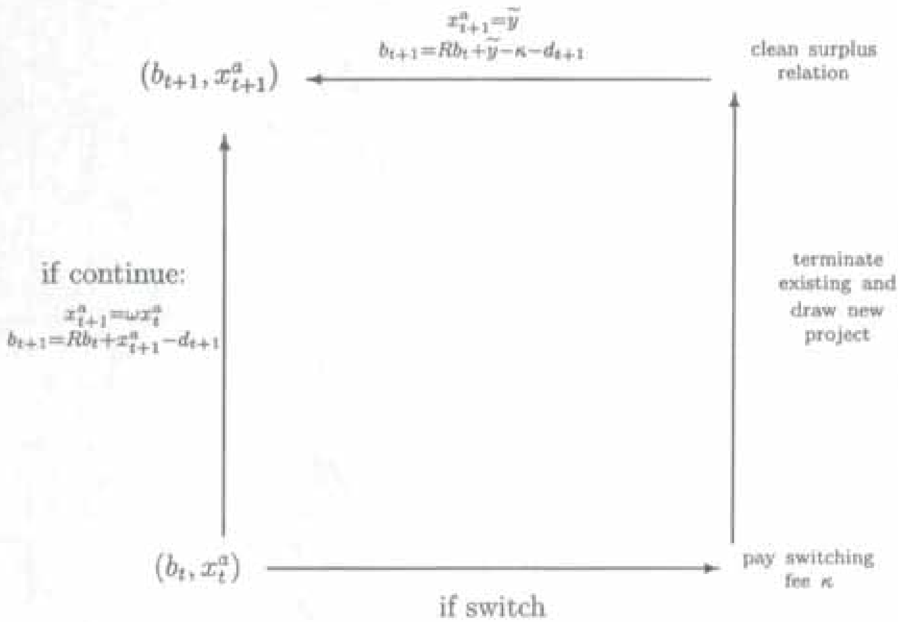
10. An alternative interpretation of eq. (8) is to give up the risk-neutral measure construction entirely, and to start with the risk-adjusted dividend discount formula. Then R is the risk-adjusted cost of capital and eq. (8) captures the idea that the long-run earnings of continuing projects revert to their risk-adjusted cost of capital. The disadvantage of this appealingly simple interpretation is that the risk-adjusted dividend discount formula is ad hoc and not grounded on no-arbitrage.

FIGURE 4



On recognizing book value b_t and abnormal one-period earnings x^a_t at date t, an adaptive firm must decide whether to continue with its existing project or switch to a new project. If it continues, the firm will realize residual earnings $x^a_{t+1} = \omega x^a_t$ in the next period. If the firm switches it (1) irreversibly abandons its existing project, (2) pays a one-time switching cost κ, (3) adopts a new project that will yield at $t + 1$ a randomly drawn residual earnings $x^a_{t+1} = \tilde{y}$. Regardless of the firm's decision at t, the firm faces a similar decision again at $t + 1$, t, $+ 2$, ad infinitem.

The clean surplus relation implies that the realization of book value is also contingent on the firm's choices:

$$b_{t+1} = \begin{cases} b^c_{t+1} \equiv Rb_t + \omega x^a_t - d_{t+1} & \text{if continue} \\ b^s_{t+1} \equiv Rb_t + \tilde{y} - \kappa - d_{t+1} & \text{if switch,} \end{cases} \tag{9}$$

where $\tilde{y}$ is the residual earning of its trial project realized in the period between t and $t + 1$, and d_{t+1} refers to potential dividends forthcoming at date $t + 1$. Equation (8) is the adaptation-adjusted version of Ohlson's LIM with errors $\tilde{\varepsilon}_{t+1}$ and $\nu_{t+1} = 0$ set to zero. Equation (9) is the adaptation-adjusted realization of the familiar clean surplus relation. I shall always assume that

$$0 < \omega < 1 < R.$$

The cumulative distribution function F of $\tilde{y}$, the new projects' unrevealed residual earnings, proxy for the quality of the project opportunities available to the firm. Note that F is the c.d.f. of $\tilde{y}$ in the equivalent risk-neutral measure. To maintain parsimony, I will assume it is independent of (b_t, x_t^a), and is stationary; that is,

$$\tilde{y} \sim F,$$

no matter how frequently the firm switches projects or how profitable the firm has been in the past. This means the supply and quality of new projects to the adaptive firm is an inexhaustible static resource.

I shall also assume that F has compact support:[11]

$$\tilde{y} \in [\underset{\sim}{y}, \bar{y}], F(\underset{\sim}{y}) = 0, F(\bar{y}) = 1.$$

The risk-neutral mean and spread[12] associated with cumulative distribution function F will be denoted

$$\mu_F \equiv E(\tilde{y}), \Delta_F \equiv \text{spread of } F,$$

where μ_F and Δ_F are risk-neutralized indicators of the quality distribution of switching opportunities available to the firm—larger μ_F and Δ_F values correspond to firms with superior adaptation opportunities. I shall focus on the case when $\underset{\sim}{y} < 0$ and $\bar{y} > 0$, and $\underset{\sim}{y} < x_t^a < \bar{y}$. Then, because replacement values of residual earnings are all drawn from F,

$$\underset{\sim}{y} \leq x_{t+\tau}^a \leq \bar{y} \quad \forall \tau \geq 0.$$

The state space probabilities associated with F correspond to the state prices implied by the firm's prospective projects' market prices. If the prospective projects are publicly traded, F is fixed uniquely and the adaptive firm is effectively a redundant security whose value is fixed by the arbitrage principle. On the other hand, if the firm's prospective projects are not publicly traded, then positing the risk-neutral CDF F is equivalent to exogenously imposing a risk-neutral measure. When the risk-neutral measure is imposed rather than set by intrinsic market forces, one is said to be in a "real options" framework rather than a bona fide equivalent risk-neutral measure. Depending on the contents of the firm's investment opportunity set, this is the main caveat for the adaptation model.

---

11. In addition to enabling us to exploit Lemma 1 in Appendix A.3, bounding the support of F will ensure the convergence of what otherwise are potentially divergent infinite series occurring in the intermediate steps of forthcoming manipulations—at least it eliminates the unpleasant task of having to prove these series are not divergent. This restriction, however, places little if any real limitation on our model since one can always think of $\underset{\sim}{y}$ and $\bar{y}$ as taking arbitrarily extreme values.

12. The spread of a distribution function F is a measure of its width. In particular, following Rothschild and Stiglitz (1970) if F_1 and F_2 are two c.d.f.'s on $[\underset{\sim}{y}, \bar{y}]$, which cross at exactly one point $\hat{y}$, then F_2 has greater spread than F_1 iff

$$F_2(y) \begin{cases} > F_1(y) & \text{if } y \in (\underset{\sim}{y}, \hat{y}); \\ = F_1(y) & \text{if } y = \hat{y}; \\ < F_1(y) & \text{if } y \in (\hat{y}, \bar{y}). \end{cases}$$

FIGURE 5

$$V(b_t, x_t^a) = \max_{c,s} \xrightarrow{\;s\;} \frac{1}{R}E[d_{t+1} + V(b_{t+1}^s, \tilde{y})|b_t, x_t^a]$$

c

$$\downarrow$$

$$\frac{1}{R}E[d_{t+1} + V(b_{t+1}^c, \omega x_t^a)|b_t, x_t^a]$$

Because an adaptive firm's optimization problem does not change from one period to the next, a recursive one-stage optimization problem captures the effect on $V(b_t, x_t^a)$ of the firm's infinite sequence of decisions; that is, $V(b_t, x_t^a$ satisfies a recursive Bellman equation at every date t. If the firm continues, its value equals the expected present cum-dividend value of a date $t + 1$ firm with abnormal earnings $x_{t+1}^a = \omega x_t^a$ and book value $b_{t+1}^c = Rb_t + \omega x_t^a - d_{t+1}$. If the firm switches, its value equals the expected discounted cum-dividend value of a date $t + 1$ firm with randomly drawn $x_{t+1}^a = \tilde{y}$ and book value $b_{t+1}^s = Rb_t + \tilde{y} - \kappa - d_{t+1}$. Accordingly, the firm continues if and only if $E[V(b_{t+1}^c, \omega x_t^a)|b_t, x_t^a] \geq E[V(b_{t+1}^s, \tilde{y})|b_t, x_t^a]$.

2.2 Optimal Adaptation: How the Firm Decides

As indicated in Figures 3 and 4, the choice faced by an adaptive firm does not change from one period to the next. Following standard dynamic programming methods, one can exploit this time translation invariance to collapse the infinite-period decision tree in Figure 3 into the one-stage (recursive) tree depicted in Figure 5.

As indicated, $V(b_t, x_t^a)$ satisfies a recursive one-stage optimization equation. If the firm continues, its value is the expected present cum-dividend value of a date $t + 1$ firm with abnormal earnings $x_{t+1}^a = \omega x_t^a$ and book value $b_{t+1}^c = Rb_t + \omega x_t^a - d_{t+1}$. If the firm switches, its value is the expected discounted cum-dividend value of a date $t + 1$ firm with randomly drawn $x_{t+1}^a = \tilde{y}$ and book value $b_{t+1}^s = Rb_t + \tilde{y} - \kappa - d_{t+1}$. Accordingly, the firm continues if and only if $E[V(b_{t+1}^c, \omega x_t^a)|b_t, x_t^a] \geq E[V(b_{t+1}^s, \tilde{y})|b, x_t^a]$; otherwise, it is advantageous for the firm to switch. Therefore, assuming the firm acts to maximize its own value, $V(b_t, x_t^a)$ satisfies[13]

$$V(b_t, x_t^a) = \frac{1}{R}\max_{\phi \in \{c,s\}} \{E[d_{t+1} + V(b_{t+1}^\phi, \omega x_t^{a\phi})|b_t, x_t^a]\}. \tag{10}$$

13. Following standard arguments for pricing derivative securities, eq. (10) is also supported by viewing the firm as a derivative security. The two primitive securities are the risk-free bond and the security that mandates a new abnormal earnings draw from the investment opportunity set every period. Here V is the value of the hedge portfolio that optimally switches back and forth between these two securities.

Equation (10) is formally an example of Bellman's equation for a class of problems known as job search problems, simple variants of which were first solved by McCall[14] (1970). The new methodological contribution of this paper is to apply the mathematical machinery developed for analyzing such stochastic control problems to the residual income valuation framework, and to exploit it for valuation. To this end, Subsection 2.3 will identify and characterize the solution of eq. (10) for $V(b_t, x_t^a)$, the adaptation-adjusted extension of Ohlson's valuation function.

I stress that even though the expression defining $V(b_t, x_t^a)$, eq. (10), does not look anything like the familiar discounted dividends formula, it is rigorously equivalent to it. The reason is that, by Bellman's original insight, eq. (10) is mathematically equivalent to[15]

$$V(b_t, x_t^a) = \max_{\substack{\text{decision}\\ \text{tree}}} \left[b_t + \sum_{\tau=1}^{\infty} R^{-\tau} E(x_{t+\tau}^a \mid b_t, x_t^a) \right], \tag{11}$$

where maximization is over the sequence of continue or switch choices in the Figure 3 decision tree subject to adaptation-adjusted LIM and the clean surplus relation, eqs. (8) and (9). Moreover, since clean surplus accounting guarantees that $b_t + \Sigma_{\tau=1}^{\infty} R^{-\tau} E(x_{t+\tau}^a | b_t, x_t^a)$ equals $\Sigma_{\tau=1}^{\infty} R^{-\tau} E(d_{t+\tau} | b_t, x_t^a)$ by the standard EBO transformation, the $V(b_t, x_t^a)$ of eq. (10) must also equal the expected present value of an adaptive firm's forthcoming dividend stream.[16]

2.3 Value of an Adaptive Firm

As described so far, the value $V(b_t, x_t^a)$ rests on three assumptions, which are nothing more than the adaptation-adjusted counterparts of the three basic assumptions of Ohlson (1995):

14. McCall studied the relationship between unemployment and the wage distribution of available jobs. Specifically, suppose an unemployed worker is periodically offered a new job as long as she remains unemployed. McCall assumes further that taking a job is irreversible and nobody is ever fired, so accepting one offer renders all trailing offers moot. Given that the worker receives unemployment compensation and has a right to refuse offers indefinitely, how high must a wage offer be to entice acceptance? Using the Bellman equation approach, McCall answered this question and derived an expression for the present value of an unemployed worker's expected income stream as a function of her wage-offer distribution. The problem I will solve for $V(b_t, x_t^a)$ is analogous to McCall's in many ways. However, adaptation is a more complex problem in that adopting a new project is not irreversible—the firm may take on a new project and then replace it later with another one.

15. Such equivalences are what make the Bellman equation approach powerful. See Stokey (1989) for an introductory discussion.

16. If this is new to the Reader, she may find it instructive to verify that Ohlson's LIM solution $V_{\text{non-adaptive}}(b_t, x_t^a) = b_t + \alpha x_t^a$ obeys the recursive equation

$$V_{\text{non-adaptive}}(b_t, x_t^a) = \frac{1}{R} E[d_{t+1} + V_{\text{non-adaptive}}(b_{t+1}^c, \omega x_t^a) | b_t, x_t^a].$$

Note that in eq. (10) $V(b_t, x_t^a)$, equals the maximum of the RHS of the latter equation and an alternative payoff. Thus, an immediate implication is that $V(b_t, x_t^a) \geq V_{\text{non-adaptive}}(b_t, x_t^a)$—consistent with Figure 1.

- the adaptation-adjusted version of LIM, eq. (8);
- the clean surplus relation, eq. (9), implied by eq. (8); and
- the Bellman equation formulation, eq. (10), of the discounted dividends formula cast in terms of residual earnings.

This subsection describes the solution of eq. (10) and discusses how it relates to Figures 1 and 2.

Expression of the formula for $V(b_t, x_t^a)$ requires introducing an integer n_F and a real number x_*^a. These numbers[17] are defined as the solutions to the two simultaneous nonlinear equations[18]

$$n_F = \left[\left[\frac{\log(\bar{y}/x_*^a)}{\log(1/\omega)}\right]\right], \tag{12}$$

and

$$\begin{aligned} x_*^a = \left[\frac{R-1}{(R-\omega)\omega}\right]\left(\frac{R^{n_{F+1}} - \omega^{n_{F+1}}}{R^{n_{F+1}} - 1}\right)\mu_F - \left(\frac{R-1}{\omega}\right)\left(\frac{R^{n_F}}{R^{n_{F+1}} - 1}\right)\kappa \\ + \left(\frac{R^{n_F} - R^{n_{F+1}}}{R^{n_{F+1}} - 1}\right)\sum_{n=0}^{n_{F+1}}\left(\frac{\omega}{R}\right)^n \int_{\underline{z}}^{\omega^{-n}x_*^a} d\bar{y}\, F(\bar{y}). \end{aligned} \tag{13}$$

The second equation is nonlinear in x_*^a because its RHS depends on n_F and x_*^a. Since these equations depend on persistence parameter ω, discount factor R, switching fee κ, and properties of distribution F, n_F, and x_*^a must vary with these model parameters.[19]

Note that x_*^a can be thought of as the adaptive firm's reservation value for residual income. This is because (as will be shown) a firm never finds it favorable to continue with a project whose existing abnormal earnings are less than x_*^a. Faced with $x_t^a < x_*^a$, the firm does better on average to switch to a new randomly drawn project. Note that $x_*^a \neq \mu_F$ because "better" is defined with respect to the present value of the firm's entire future earnings stream, not just its next-period abnormal earnings.

A firm with residual income $x_t^a > x_*^a$ will not find it favorable to immediately switch projects. In fact, since abnormal earnings evolve according to $x_{t+1}^a = \omega x_t^a$ during the continuation period, the firm continues for another

$$\tau = \left[\left[\frac{\log(x_t^a/x_*^a)}{\log(1/\omega)}\right]\right]$$

17. The technical origin of these numbers and their defining expressions is motivated below and in the proof of Lemma 1 in Appendix A.2.

18. Notation: [[z]] refers to the smallest integer greater than or equal to z (e.g., [[z]] is the value of z rounded off upward to the nearest integer).

19. While these long expressions make the forthcoming comparative statics analysis rather tedious, there is no reason to be put off by them; an interest in analyzing such formulas is *not* necessary for following the rest of this article.

periods. This is because at date $t + \tau$, the abnormal earnings x^a_{t+1} becomes[20]

$$x^a_{t+\tau} = \omega^\tau x^a_t = \exp\left[-\tau \log(1/\omega)\right]x^a_t \le \exp\left[-\frac{\log(x^a_t/x^a_*)}{\log(1/\omega)}\log(1/\omega)\right]x^a_t = x^a_*,$$

so switching becomes favorable. In other words, because the residual income of all projects invariably decays to zero, all adaptive firms eventually find switching projects favorable—it is just a matter of time. The only difference between having a high current x^a_t and a smaller one is the number of periods τ the adaptive firm will put off switching. Since $\bar{y}$ is the largest possible value of x^a_t, n_F is the largest number of continuation periods for a new project.

The formula given above for τ implies that all firms with residual income exceeding x^a_*/ω^{n-1} but less than x^a_*/ω^n will continue for exactly $\tau = n$ periods. This means if one defines

$$y_n \equiv \begin{cases} \underline{y} & \text{if } n = -1, \\ \omega^{-n}x^a_* & \text{if } 0 \le n < n_F, \\ \bar{y} & \text{if } n = n_F, \end{cases}$$

then the entire range $[\underline{y}, \bar{y}]$ of possible x^a_t values can be partitioned into $n_F + 1$ subsegments

$$\mathcal{S}_n \equiv [y_{n-1}, y_n] \quad \forall\, n \in \{0, \ldots, n_F\}.$$

An adaptive firm with earnings $x^a_t * \mathcal{S}_n$ continues for exactly $\tau = n$ periods before switching.[21]

The adaptation-adjusted valuation function $V(b_t, x^a_t)$ directly reflects this foliation of $[\underline{y}, \bar{y}]$:

Lemma 1 *If $x^a_t \in \mathcal{S}_n$ for any $n \in \{0, \ldots, n_F\}$, the unique continuous solution to eq.* (10) *is*

$$V(b_t, x^a_t) = b_t + \left(\frac{\omega}{R - \omega}\right)\begin{cases} \gamma_0 x^a_* + \lambda_0 x^a_t & \text{if } x^a_t \in \mathcal{S}_0, \\ \gamma_1 x^a_* + \lambda_1 x^a_t & \text{if } x^a_t \in \mathcal{S}_1, \\ \gamma_2 x^a_* + \lambda_2 x^a_t & \text{if } x^a_t \in \mathcal{S}_2, \\ \vdots & \\ \gamma_{n_F} x^a_* + \lambda_{n_F} x^a_t & \text{if } x \in \mathcal{S}_{n_F}, \end{cases} \tag{14}$$

where

$$\gamma_n = \left(\frac{R - \omega}{R - 1}\right)R^{-n}, \quad \lambda_n = 1 - \left(\frac{\omega}{R}\right)^n. \tag{15}$$

20. If the evolution x^a_{t+1} suffers noise ($\tilde{\varepsilon}_{t+1}$) or exogenous information (ν_{t+1}) disturbances, the simple argument here must be modified. Maintaining this simple argument and its ultimate consequence that $V(b_t, x^a_t)$ is piecewise linear in x^a_t (Lemma 1) is why $\tilde{\varepsilon}$ and ν are set to zero in this article. The analysis when $\tilde{\varepsilon}$ and ν fluctuations are permitted is more complicated because the number of continuation periods in effect becomes a random number.

21. I use the convention that an $x^a_t = y_n$ firm, which would be indifferent between continuing for n or $n + 1$ periods, switches after n periods.

Consequently $V(b_r, x_r^a)$ is a continuous, increasing, convex function in x_r^a.

The proof of this Lemma is given in Appendix A.2.

Lemma 1 reveals that $V(b_r, x_r^a)$ is a remarkably simple function—a piecewise linear continuous function with n_F slope-shifting kinks at $x_r^a = \omega^{-n} x_*^a$ ($n = 0, \ldots, n_F - 1$). While $V(b_r, x_r^a)$ is linear between kinks, its slope changes discretely at each of these kinks. These slope changes embody the effect of adaptation on firm value. Without adaptation $V(b_r, x_r^a)$ would simply fall linearly with x_r^a all the way down to b_r at $x_r^a = 0$. But due to slope changes as x_r^a falls through each kink, $V(b_r, x_r^a)$ flattens out as x_r^a approaches zero and stays above b_r.

It is no accident that kinks are also where the continuation time τ defined earlier changes. As long as x_r^a does not fall enough to change τ, $V(b_r, x_r^a)$ varies linearly with x_r^a. But if x_r^a falls through a kink, the firm is able to reduce the continuation time of a project accordingly. This reduction ameliorates the damage to $V(b_r, x_r^a)$ so that the resulting change in $V(b_r, x_r^a)$ is less than linear. In summary because of the firm's ability to optimally time the discontinuation of its projects and switch to new ones, $V(b_r, x_r^a)$ is convex in x_r^a.

Note γ_n and λ_n determine the functions $\gamma(x_r^a)$ and $\lambda(x_r^a)$ introduced in eq. (5): if $x_r^a \in S_n$, then

$$\gamma(x_r^a) = \gamma_n, \quad \lambda(x_r^a) = \lambda_n.$$

It is easy to verify that the properties claimed for $\gamma(x_r^a)$ and $\lambda(x_r^a)$ in Section 1 are satisfied. In particular, $\gamma(0) > 0$ so that $V(b_r, 0) - b_r > 0$. In the opposite direction, $\gamma(\infty) = 0$ and $\lambda(\infty) = 1$ so that

$$\lim_{x_r^a \to \infty} V(b_r, x_r^a) \to b_r + \alpha x_r^a.$$

The first proposition concerns the valuation coefficients A, B, and C introduced earlier in eq. (6).

Proposition 1: *The linear book value and earnings valuation model, eq. (6), incurs the following values for book value valuation coefficient B, earnings coefficient C, and remainder A for an adaptive firm:*

$$A = \left(\frac{\omega}{R - 1}\right) R^{-n} x_*^a,$$

$$B = 1 - \left(\frac{R - 1}{R - \omega}\right)\left[1 - \left(\frac{\omega}{R}\right)^n\right]\omega,$$

$$C = \left(\frac{R\omega}{R - \omega}\right)\left[1 - \left(\frac{\omega}{R}\right)^n\right],$$

where integer n is the value of n, which accommodates

$$x_t \in \left[\frac{y_{n-1} + (R - 1)(b + d)_t}{R}, \frac{y_n + (R - 1)(b + d)_t}{R}\right].$$

Since n increases with increasing x_t, A, B, and C vary (via n) with earnings x_t as illustrated in Figure 2 in Section 1.5. As depicted, B increases while C falls as $x_t \mapsto 0$.

As advertised, B and C exhibit complementary behavior as x_t approaches zero. If earnings are small, $n \mapsto 0$. In this limit book value coefficient $B \mapsto 1$ while earnings x_t coefficient $C \mapsto 0$. These predictions agree with the empirical findings of (among others) Beaver, Barth, and Landsman and Burgstahler and Dichev that the earnings coefficient is markedly smaller for earnings-poor firms.

In the other extreme, if x_t is large then n is also large. In this limit the coefficients recover their earnings-independent Ohlson values listed in eq. (7). Thus, $\hat{V}(b_t, x_t)$ becomes linear in x_t for earnings-rich firms.

In the sense of being the (only) value component not directly proportional to b_t or x_t, A is the residue term of $\hat{V}(b_t, x_t)$. And like B and C, A is not constant; it depends on earnings and book value as well as all the other exogenous model parameters (some indirectly via x_t^a). As such, the value of A is firm-specific and fluctuates idiosyncratically across firms. Therefore, A potentially *looks* like an omitted-variable or ν term in cross-sectional valuation studies. In other words, the value of an adaptive firm may exhibit an apparent ν component, A, even if the underlying information dynamics has no ν.

In the large earnings limit residue $A \mapsto 0$. It is in the small earnings limit that A assumes its largest values. The next section describes an intuitive interpretation for residue A when earnings are small.

2.4 Dormant Value

A gauge of the impact of adaptability on the relation $V(b_t, x_t^a)$ between firm value and accounting numbers is the difference between V and the corresponding (Ohlson) value implied when the firm is committed forever to the existing project:

$$\mathrm{IOA}(x_t^a) \equiv V(b_t, x_t^a) - (b_t + \alpha x_t^a),$$

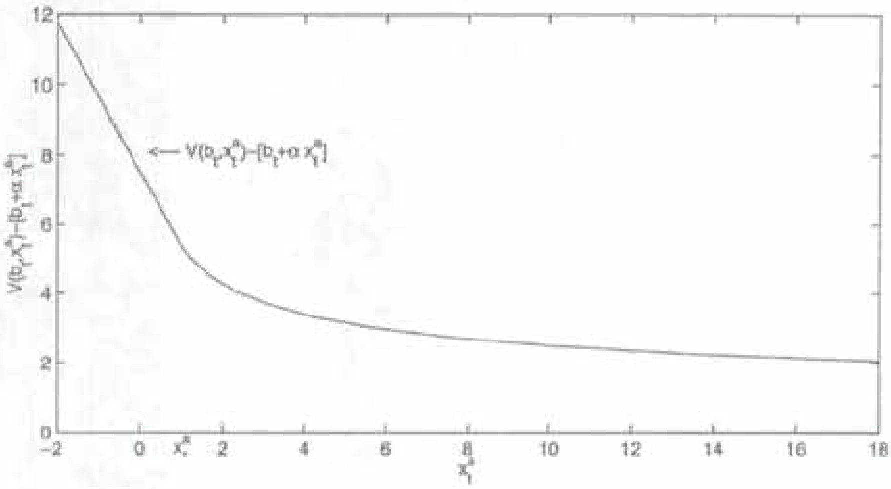
where IOA is the incremental impact of adaptation on the value relation relative to that of an otherwise identical nonadaptive firm irrevocably committed to its existing project.[22]

The next lemma gives the formula for IOA, whose dependence on x_t^a is plotted in Figure 6.

Lemma 2 *The impact of adaptability is greatest for residual earnings-poor firms and strictly decreases with increasing x_t^a according to:*

22. IOA equals the value of the "real" perpetual switching options held by an adaptive firm only if the firm's realized exercise policy does not shift the original Martingale measure and the discount factor R.

FIGURE 6



Alternative representation of Figure 1 which highlights the nonlinear component of V. Adaptability provides firms an inexhaustible stream of options to abandon and replace unprofitable projects with projects which, on average, meet industry profitability standards. The positive difference between $V(b_t, x_t^a)$ and the nonadaptive-firm value $b_t + \alpha x_t^a$ measures the impact of adaptability (IOA) on the relation between V and accounting numbers. As depicted IOA is schematically reminiscent of a perpetual put option value: for the price of the existing residual income stream (plus switching cost) the firm may draw a new project any and every period.

$$\mathrm{IOA}(x_t^a) = \alpha\left\{\left(\frac{R-\omega}{R-1}\right)x_*^a - \omega^n x_t^a\right\}R^{-n}\ \forall\ x_t^a \in S_n.$$

This lemma follows directly from the definition of IOA and Lemma 1.

According to this formula, which assumes unbiased accounting,[23] an adaptive firm has value exceeding b_t even when its residual earnings are zero and its accounting is unbiased. In fact, IOA assumes its greatest value when abnormal earnings x_t^a is smallest ($x_t^a \in S_0$). As a benchmark, consider what in Section 1.6 was referred to as dormant value:

$$\mathrm{IOA}(0) = \left(\frac{\omega}{R-1}\right)x_*^a = A_1 \quad (16)$$

Dormant value IOA(0) is proportional to x_*^a, the minimum value of abnormal earnings an adaptive firm requires to continue with a project. As indicated by eq. (13), x_*^a depends on values of the exogenous parameters characterizing the firm, so it is

23. Recall that "unbiased accounting" here means the LIM evolution of x_t^a does not depend explicitly on b_t.

firm-specific. How x_t^a relates to the exogenous parameters will be discussed in Section 2.6.

2.5 Adaptation Frequency

Better opportunities mean that new projects are, on average, more profitable and deserving of continuation for more periods. Simultaneously, the presence of better opportunities makes switching more attractive and so raises the continuation threshold for ongoing projects. On balance how does the adaptation rate vary with the quality distribution of available project opportunities?

A simple calculation reveals the optimal balance. Following the discussion earlier of the markers $\{y_n\}$, the probability of a new draw having a lifespan of $\tau = n$ is

$$F(y_n) - F(y_{n-1}).$$

Then since $F(y_{n_F}) = F(\bar{y}) = 1$ the average duration of a new project is

$$\begin{aligned} E(\tau) &= \sum_{n=1}^{n_F} n[F(y_n) - F(y_{n-1})] \\ &= n_F F(\bar{y}) + \sum_{n=0}^{n_F-1} [n - (n+1)]F(y_n) \\ &= n_F - \sum_{n=0}^{n_F-1} F(y_n). \end{aligned}$$

Since $n_F = \Sigma_{n=0}^{n_F-1} 1$, E(τ) simplifies to:

Lemma 3 *The average project lifespan—the number of periods an adaptive firm continues with a new project before switching—is*

$$E(\tau) = \sum_{n=0}^{n_F-1} \{1 - F(y_n)\}.$$

As given, $E(\tau)$ depends on F and, via y_n and n_F, the reservation value of abnormal earnings x_t^a and abnormal earnings persistence ω. Since F is a cumulative distribution function, $\partial F(y_n)/\partial x_t^a > 0$. Moreover, by eq. (12), $\partial n_F/\partial x_t^a < 0$. Therefore, applying the chain rule to $E(\tau)$ yields

$$\left.\frac{\partial E(\tau)}{\partial x_t^a}\right|_{\omega, F} < 0.$$

This hints at the answer to the question posed at the start of this section: all else equal firms with a greater abnormal earnings threshold, x_t^a, have projects with shorter lifespans and switch projects more frequently. However, to address this question more thoroughly, it is necessary to determine the responses of x_t^a and $E(\tau)$ to perturbations of the other exogenous parameters. The next section turns to this task.

To suppress boundary effects in this section I shall assume $\bar{y}$ is extremely large. In this limit $n_F \mapsto \infty$ and eq. (13) simplifies to

$$x_*^e = \left[\frac{R-1}{(R-\omega)\omega}\right]\mu_F - \left(\frac{R-1}{\omega R}\right)\kappa + \left(\frac{R-1}{R^2}\right)\sum_{n=0}^{\infty}\left(\frac{\omega}{R}\right)^n \int_{\underline{y}}^{\omega^{-n}x_*^e} d\tilde{y}\, F(\tilde{y}). \quad (17)$$

2.6 Comparative Statics Results

Starting from eq. (17) it is easy to see, with the help of Lemma 1 in Appendix A.3, that increasing the spread Δ_F of F increases x_*^e. In particular, taking the partial derivative with respect to Δ_F of eq. (17) and simplifying yields

$$\left[1 - \left(\frac{R-1}{R^2}\right)\sum_{n=0}^{\infty} R^{-n} F(\omega^{-n}x_*^e)\right]\frac{\partial x_*^e}{\partial \Delta_F}\bigg|_{\substack{\omega,R,\\ \kappa,\mu_F}}$$

$$= \frac{R-1}{\omega R}\sum_{n=0}^{\infty}\left(\frac{\omega}{R}\right)^n \frac{\partial}{\partial \Delta_F}\bigg|_{\substack{x_*^e,\omega,\\ R,\kappa,\mu_F}} \int_{\underline{y}}^{\omega^{-n}x_*^e} d\tilde{y}\, F(\tilde{y}).$$

Since $F(\tilde{y}) \leq 1$, the bracketed factor on the LHS is strictly positive because

$$\left(\frac{R-1}{R^2}\right)\sum_{n=0}^{\infty} R^{-n}F(\omega^{-n}x_*^e) \leq \left(\frac{R-1}{R^2}\right)\sum_{n=0}^{\infty} R^{-n} = \frac{1}{R} < 1.$$

The RHS expression is also positive since, following Appendix Lemma 1,

$$\frac{\partial}{\partial \Delta_F}\bigg|_{\substack{x_*^e,\omega,\\ R,\kappa,\mu_F}} \int_{\underline{y}}^{\omega^{-n}x_*^e} d\tilde{y}\, F(\tilde{y}) \geq 0.$$

Consequently,

$$\frac{\partial x_*^e}{\partial \Delta_F}\bigg|_{\omega,R,\kappa,\mu_F} > 0.$$

Moreover, eq. (12) then implies that n_F decreases with the spread Δ_F.

Figure 7 tabulates the implications of the entire comparative statics analysis. As discussed in Appendix A.2, the proofs that

$$\frac{\partial x_*^e}{\partial \mu_F}\bigg|_{\omega,R,\kappa,\Delta_F} \geq 0, \quad \frac{\partial x_*^e}{\partial \kappa}\bigg|_{\omega,R,F} < 0$$

are analogous to the derivation of

$$\frac{\partial x_*^e}{\partial \Delta_F}\bigg|_{\omega,R,\kappa,\mu_F} \geq 0.$$

However, the associations of x_*^e with ω and R are less transparent and may vary depending on the magnitudes of the exogenous parameters.

FIGURE 7

parameter	κ	ω	R	μ_F	Δ_F
x_*^a	$-$	$-^*$	$-^\dagger$	$+$	$+$
n_F	$+$	$+^\dagger$	$+^\dagger$	$-$	$-$
$E[\tau]$	$+$	$+^\dagger$	$+^\dagger$	$-^\dagger$	$+^*$
dormant value $IOA(0)$	$-$	$+^\dagger$	$-^\dagger$	$+$	$+$
$V^{\text{non-adaptive}}(b_t, x_t^a)$	0	$+$	$-$	0	0

Signs predicted by comparative statics for the association between the exogenous parameters and some of the endogenous valuation measures. Superscript "†" indicates the signs are proven only in the special case where F is of the form given in Eq. (18). In contrast, unsuperscripted entries are shown to hold without qualification. Superscript "*" indicates that while the listed sign prevails most of the time, the sign may flip for extreme values of the exogenous parameters. In particular, suppose F is given by Eq. (18). Then $(\partial x_*^a/\partial\omega) < 0$ when ω is sufficiently less than 1, but $(\partial x_*^a/\partial\omega) > 0$ as $\omega \mapsto 1^-$ and $R \mapsto 1^+$. Similarly $[\partial E(\tau)/\partial\Delta_F] \geq 0$ when $\mu_F >> \kappa$, but $[\partial E(\tau)/\partial\omega] < 0$ when κ is comparable to or larger than μ_F. Associations indicated for $V^{\text{non-adaptive}}(b_t, x_t^a) \equiv b_t + \alpha x_t^a$ hold for all values of (b_t, x_t^a). A 0 indicates no relation.

To get a handle on the response of x_*^a and the other endogenous quantities to variations of ω and R, it is instructive to work out a tractable example. To this end, consider the special case when[24]

$$F(\tilde{y}) = \begin{cases} 0 & -\infty < \tilde{y} < \mu_F - \frac{1}{2}\Delta_F \\ \frac{1}{2} & \mu_F - \frac{1}{2}\Delta_F \leq \tilde{y} < \mu_F + \frac{1}{2}\Delta_F \\ 1 & \mu_F + \frac{1}{2}\Delta_F \leq \tilde{y} < +\infty. \end{cases} \tag{18}$$

24. A firm with this F has half a chance of drawing $\tilde{y} = \mu_F + \frac{1}{2}\Delta_F$ and half a chance of drawing $\tilde{y} = \mu_F - \frac{1}{2}\Delta_F$ when it switches projects.

In this special case, eq. (13) reduces to

$$\left[\Sigma(\frac{1}{R}, \hat{M}_+) + \Sigma(\frac{1}{R}, \hat{M}_-)\right]\omega x_t^a = \left[\Sigma(\frac{\omega}{R}, \hat{M}_+) + \Sigma(\frac{\omega}{R}, \hat{M}_-)\right]\mu_r - 2\kappa + \frac{1}{2}\left[\Sigma(\frac{\omega}{R}, \hat{M}_+) - \Sigma(\frac{\omega}{R}, \hat{M}_-)\right]\Delta_r,$$

where

$$\Sigma(x, n) \equiv \sum_{m=0}^{n} x^m = \frac{1 - x^{n+1}}{1 - x},$$

and $\hat{M}_\pm$ are the smallest integers such that markers $y_{\hat{M}_\pm} \geq \mu_r \pm \frac{1}{2}\Delta_r$:

$$\hat{M}_\pm \equiv \left[\left[\log\left(\frac{\mu_r \pm \frac{1}{2}\Delta_r}{x_t^a}\right) / \log\left(\frac{1}{\omega}\right)\right]\right].$$

In addition, the expected continuation time given in Lemma 3 reduces to

$$E(\tau) = \frac{1}{2}(\hat{M}_+ + \hat{M}_-).$$

Performing comparative statics on these special-case expressions yields the daggered (†) entries in Figure 7.

The remainder of the Figure 7 entries are derived in Appendix A.2. While the daggered entries of Figure 7 may prevail beyond the special quality distribution, it is important to keep in mind that they are derived assuming eq. (18). Our final proposition highlights some of the empirical implications of Figure 7.

Proposition 2: Following Figure 7:

- *Higher switching cost κ reduces an adaptive firm's reservation value of abnormal earnings, its switching frequency, and its dormant value.*
- *While higher abnormal earnings persistence ω reduces an adaptive firm's reservation value of abnormal earnings and its switching frequency, it enhances its dormant value.*
- *Larger discount rate R reduces an adaptive firm's reservation value of abnormal earnings, its switching frequency, and its dormant value.*
- *Higher mean quality μ_r of switching opportunities increases an adaptive firm's reservation value of abnormal earnings, switching frequency, and its dormant value.*
- *Greater spread Δ_r in switching opportunities increases an adaptive firm's reservation value of abnormal earnings and its dormant value.*

While most of these results escape the grasp of naive intuition, it is easy to rationalize them in hindsight. Higher quality switching opportunities encourage switching and, accordingly, are associated with a larger reservation value of abnormal earnings,[25] a higher switching frequency, and greater adaptation value. On the flip side, switching cost discourages switching and has exactly the opposite effects.

The effects of ω are slightly more subtle. Greater persistence tends to magnify the correlation between existing and prospective abnormal earnings numbers. Accordingly, when ω is bigger, the option to replace low earnings with higher ones is more valuable. Therefore, greater ω enhances the impact of adaptability. On the other hand, while larger ω increases, the value of the *option* to switch, the *act* of switching is discouraged since a higher ω means an adaptive firm gives up more value when it terminates a project with a positive abnormal earnings number. Hence higher persistence dampens enthusiasm for switching and reduces the reservation value of abnormal earnings and switching frequency.

A larger discount rate increases myopia by enhancing the significance of existing earnings at the expense of the longer-term earnings stream. Faced with less emphasis on longer-term income, firms pay attention to next period's earnings at the expense of what lies beyond. This means immediate earnings has enhanced significance. In this (somewhat counterintuitive) sense increasing R has the same effect as increasing ω: reduction of the reservation value of abnormal earnings and the switching frequency. On the other hand, dormant value falls with increasing R as intuitively expected: greater R increases the discount on the value of an adaptive firm's options to switch in the future.

3. Additional Remarks

This paper puts forth a formal framework for thinking about the valuation consequences of abandonment and adaptation, and for relating the impact of adaptation to exogenous characteristics such as abnormal earnings persistence, the discounting rate, switching costs, and the quality distribution of a firm's opportunity set. As it is, this paper makes two contributions to the securities valuation literature. First it incorporates adaptation into the Ohlson framework and derives the adaptation-influenced valuation coefficients. As found, valuation function $V(b_t, x_t^a)$, depicted in Figure 1, is convex in earnings—in agreement with Burgstahler and Dichev. Moreover, as depicted in Figure 2, the valuation coefficients of book value and earnings vary in a complementary way. Additionally, the model suggests that adaptation generates an idiosyncratic residual term A—the dormant value—which mimics the contribution of Ohlson's ν in LIM models. In summary, the model presented here establishes a formal stylistic theoretical framework for thinking about the valuation implications of adaptation.

This paper's second contribution is its exploitation of the Bellman equation. By highlighting the notion of abnormal earnings dynamic *contingency* and solving

25. A larger reservation value of abnormal earnings means a firm has a higher standard for abnormal earnings—it needs a higher value of abnormal earnings to justify continuing with a project.

the Bellman equation for $V(b_t, x_t^a)$, this paper illustrates how dynamic programming may be useful for fundamental securities analysis. Judging from the expanding body of evidence on other nonlinear relationships between market values and accounting numbers (e.g., Kinney et al. [1999]; Easton [1999]), I would be surprised if the model presented herein is anything more than a modest beginning. Future work, starting with Yee (1999), will explore contingent information dynamics in the context of limited liability, bankruptcy, liquidation, dividend policy, and taxation. Preliminary calculations indicate these constructs can be incorporated into the residual income valuation framework using the methodology espoused in this article.

The wealth of incoming empirical valuation studies in a variety of diverse institutional contexts undoubtedly will create a market for guidance from more sophisticated theoretical models. Current areas of interests include: how rate of return and incentive regulation affects the valuation relation (e.g., Blacconiere et al. [1999]; Nwaeze [1998]); how the anticipated taxation curve impacts firm value (e.g., Collins and Kemsley [1999]; Harris and Kemsley [1999]); how limited liability influences the significance of reported earnings (e.g., Fischer and Verrecchia [1997]); and why Internet companies are valued by the market as they are.

Thinking ahead, nonlinear valuation theory may also contribute to a better understanding of how securities valuation is impacted by features of the corporate form, stakeholder pecking order and property rights, and macroeconomic network externalities. While these topics have attracted attention off and on,[26] in my opinion they have never been adequately pursued. The lack of institutionally correct valuation models appears to have substantially impaired the intellectual credibility of fundamental securities analysis.

So-called "intrinsic" value invariably contains an extrinsic[27] or macroeconomic network component: a firm's stock price determines its ability to make subsidiary acquisitions as well as raise new capital for internal projects. Therefore, a firm's stock price—as determined by market sentiment—and its intrinsic value are mutually reinforcing. Market sentiment may become a self-fulfilling prophecy if a firm's inflated or deflated market price impacts its ability to raise capital and pursue investment opportunities.[28]

In this regard, I see no reason why securities analysis should start with a premise that value is somehow inherently intrinsic. Taking it a step further, the salient ingredient of an Internet valuation model is probably not hard to identify: a path-dependent self-reinforcing interplay between securities price, capital and

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26. For instance, see Gamble (1986), Samuelson (1996), Toms (1998), and Watts and Zimmerman (1979). Ohlson and Zhang (1998) also suggested studying the relation between accounting and property rights.

27. Indeed, the dividend discount formula is only well defined with the context of a market equilibrium with a state price vector.

28. In a different setting, Yee (1997) discusses how otherwise independent drivers might mutually reinforce an outcome which would not occur except in a coevolutionary context.

market niche formation, and network externalities. I am optimistic that nonlinear modeling approaches such as the one presented herein illustrate tools useful for pursuing such valuation models for securities analysis.

APPENDIX A

Appendix A.1 MM, Convexity, and Complementarity: An Equivalence Theorem

This section introduces the definitions used herein of Modigliani-Miller invariance (MM), earnings convexity, and the two versions of complementarity (earnings complementarity and book value complementarity). As explained, these concepts are *not* independent. The *Equivalence Theorem* (Theorem 4) is the central result of this section. It says that if MM holds, then earnings convexity is equivalent to both versions of complementarity in *any* accounting system satisfying certain mild technical conditions.

Since its MM invariance requirement is not always satisfied, the Equivalence Theorem does not apply to many situations of interest. Specifically, it does not apply to models with informative dividends or tax frictions.[29] Moreover, since the equity component of a leveraged firm with defaultable debt is not MM invariant, the Equivalence Theorem does not apply to leveraged equity. Nonetheless, the Equivalence Theorem does govern the behavior of many linear and nonlinear valuation functions, including the Ohlson and Feltham-Ohlson models and the adaptation model presented in this article.

In addition to MM invariance, the Equivalence Theorem relies on the use of a "good" accounting system. Rather than descend into the rich technicalities of what constitutes an acceptable accounting system, here I will simply introduce a much stronger assumption: Markovian accounting. An accounting system is Markovian if value function[30]

$$V = V(b_t, d_t, x_t^a, \nu_t)$$

is fully characterized by the contemporaneous accounting variables and exogenous information proxy ν_t. Rewriting eq. (4) as

$$V(b_t, d_t, x_t^a, \nu_t) = b_t + \sum_{\tau=1}^{\infty} R^{-\tau} E(x_{t+\tau}^a \mid b_t, d_t, x_t^a, \nu_t)$$

reveals that an accounting system is Markovian if the abnormal earnings dynamic is Markovian. Examples of Markovian accounting systems include Ohlson (1995); Feltham and Ohlson (1995) with ν_t expanded to incorporate operating assets; eq. (8) in the main text here; and, more broadly, any efficient (Ohlson [1999]) accounting system. It turns out that Markovian accounting is a sufficient but *not* necessary assumption. However, since virtually all models of contemporary interest satisfy Markovian accounting, and since relaxing it

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29. A discussion which does justice to the subtle but powerful role of MM invariance in securities valuation is beyond the scope of this appendix. An attempt is made by Yee (1999).

30. I will also tacitly assume that V is differentiable. However, differentiability is an assumption of convenience, not necessity. As long as V is suitably well-behaved (and I cannot imagine any valuation function of empirical relevance which would not be), this discussion extends to nondifferentiable V's (such as the piecewise differentiable adaptation solution) with differentiation replaced by finite differencing.

significantly complicates the formulation of the Equivalence Theorem, I shall assume Markovian accounting here. A more general version of the Equivalence Theorem in non-Markovian accounting systems is given in Yee (1999).

Before defining MM invariance, it is useful to specify exactly what is meant by the relevant partial derivatives:

$$\frac{\partial d_t}{\partial d_t} = 1, \quad \frac{\partial x_t}{\partial d_t} = 0, \quad \frac{\partial v_t}{\partial d_t} = 0,$$

$$\frac{\partial d_t}{\partial b_t} = 0, \quad \frac{\partial x_t}{\partial b_t} = 0, \quad \frac{\partial v_t}{\partial b_t} = 0,$$

$$\frac{\partial b_s}{\partial d_t} = \begin{cases} 0 & \text{if } s = t \\ 0 & \text{if } s < t, \end{cases} \qquad \frac{\partial b_s}{\partial b_t} = \begin{cases} 0 & \text{if } s = t \\ 0 & \text{if } s < t, \end{cases}$$

where all uninvolved variables are held fixed in the partial derivatives, and v represents an exogenous information variable influencing the abnormal earnings dynamic. These assumptions imply

$$\frac{\partial x_t^a}{\partial d_t} = 0, \quad \frac{\partial x_t^a}{\partial b_t} = 0.$$

I will be agnostic regarding the partial derivatives of noncontemporaneous variables. A dividend or book value change can impact future earnings, book value, and dividends in state contingent ways because accrued dividends might earn interest as well as possible abnormal returns.

To express the idea that dividends displace book value dollar-for-dollar, Ohlson (1995) writes $\partial b_t/\partial d_t = -1$. In contrast, above I have declared that $\partial b_t/\partial d_t = 0$. My deviation from Ohlson is only in notation, not substance. In my construction, the contemporaneous information set,

$$\Omega_t^a = \{b_t, d_t, x_t^a, v_t\},$$

is thought of as being comprised of independent variables. Accordingly, in the Ω_t^a "coordinate system" $\partial/\partial d_t$ means the derivative with respect to d_t with *all* other contemporaneous variables—including b_t—held fixed. Hence $\partial b_t/\partial d_t = 0$.

Ohlson's notion of dividend displacement is captured in my notation by introducing the *total* dividend derivative D_t:

$$D_t(V) = -\frac{\partial V}{\partial b_t} + \frac{\partial V}{\partial d_t}.$$

Here D_t captures the idea that an increase in dividends increases d_t dollar-for-dollar and decreases b_t commensurately; D_t checks for *cum*-dividend dividend invariance in the sense that dividend displacement is equivalent to $D_t(d_t + V) = 0$.

Also needed is a way to probe for *ex*-dividend dividend invariance. To this end, introduce the operator[31]

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31. The operator I_t assumes different representations in different coordinate systems (Yee [1999]). For instance, when operating on functions of an earnings-based coordinate system, $\Omega_t = \{b_t, d_t, x_t, v_t\}$,

$$I_t(V) = \frac{\partial V}{\partial d_t}.$$

Note that I_t checks for whether the preceding dividend payment has a remnant impact on V: it does iff $I_t(V) \neq 0$, which would indicate d_t has signaling or other information value.

In any Markovian accounting system, a firm is said to be Modigliani-Miller dividend invariant if and only if its value V satisfies:[32]

$$D_t(d_t + V) = 0 \tag{A.1}$$

and

$$I_t(V) = 0. \tag{A.2}$$

Condition (A.1) is equivalent to Ohlson's dividend displacement property, which in my notation is $D_tV = -1$. Condition (A.2) means that V has no recall of d_t; that is, dividend payment d_t bequeaths V with no lasting impact beyond dividend displacement. Conditions (A.1) and (A.2) are independent; there are models satisfying (A.1) but not (A.2) and vice versa. There are also models which violate both (Yee [1999]).

Together, conditions (A.1) and (A.2) are consistent with the Modigliani-Miller formulation of MM invariance (e.g., Miller [1988]) in a Markovian setting. Modigliani and Miller show that whenever investors can reverse a firm's financing arrangements with independent arbitrage, a firm's total value is independent of its capital structure. In other words, a firm's (perceived) value V equals its accrued wealth plus its (perceived) ability to produce additional wealth. Since its productivity stems from investment and not financing policy, capital structure does not matter if it does not affect (perceived) investment policy. Moreover, since a dividend payment is nothing but an implementation of financing policy, a corollary of capital structure independence is dividend invariance:

> If dividend (wealth distribution) policy has no bearing on a firm's perceived ability to create wealth, then dividend payments cannot affect V beyond simple dollar-for-dollar displacement. Hence, given an investment policy, a firm's cum-dividend total[33] value is dividend invariant.

As elegantly pointed out by Ohlson, the MM insight of separating wealth distribution from wealth creation occurs naturally within the residual income framework (Ohlson [1995]).

$$I_t(\hat{V}) = \frac{\partial \hat{V}}{\partial d_t} + \left(\frac{R-1}{R}\right)\frac{\partial \hat{V}}{\partial x_t}.$$

Since I_t has varying representations over different coordinate systems, it is fruitful to think of it as having a conceptual existence beyond its coordinate-dependent manifestations.

32. If an accounting system is non-Markovian, one is faced with the prospect of constructing representations of multitemporal dividend policy invariance, which is beyond the scope of this Appendix (see Yee [1999]).

33. It is important to note that capital structure and dividend invariance apply only to total firm value—not to the value of the individual equity or debt slices of a leveraged firm. Indeed, whenever bondholders face a nonzero chance of default, a dividend payment reallocates wealth from bondholders to equityholders. In such cases, upon a dividend payment the cum-dividend value of equity rises while the value of debt decreases. Only the total cum-dividend value of equity plus debt is MM invariant.

The key is to identify created wealth with abnormal earnings, and accrued wealth with book value.[34] Then the MM proposition translates into:

> A firm is MM invariant if its abnormal earnings dynamic is unaffected by its dividend policy.

The most general function satisfying condition (A.1) is

$$V(b_t, d_t, x_t^a, \nu_t) = b_t + \Phi\,[(b + d)_t, x_t^a, \nu_t].$$

Imposing condition (A.2) on this function yields the following truism:

Theorem 1 *In any Markovian accounting system MM dividend invariance implies the following:*

$$V = V(b_t, x_t^a, \nu_t)$$
$$V = b_t + v(x_t^a, \nu_t).$$
$$\textstyle\sum_{\tau=1}^{\infty} R^{-\tau} E\,(x_{t+\tau}^a | b_t, d_t, x_t^a, \nu_t) \text{ does not depend on } b_t \text{ or } d_t.$$

Proof: Solving the two first order differential eqs. (A.1) and (A.2) yields the second expression, $V = b_t + v(x_t^a, \nu_t)$. In turn, this implies the first expression. Equivalence of the second and third expressions follows from the EBO formula, eq. (4).

QED

In the remainder of this section, I shall assume that the abnormal earnings dynamic is Markovian. By Theorem 1, this means $V = V(b_t, x_t^a, \nu_t)$. Since $x_t^a = Rx_t - (R - 1)(b + d)_t$ by the clean surplus relation and the definition of x_t^a, transforming to an earnings x_t coordinate system

$$\Omega_t \equiv \{b_t, d_t, x_t, \nu_t\}$$

yields

$$\hat{V}(b_t, d_t, x_t, \nu_t) \equiv V[b_t, Rx_t - (R - 1)(b + d)_t, \nu_t].$$

The second expression in Theorem 1 then immediately implies:

Corollary 1 *In any Markovian accounting system MM invariance implies there exists some function v such that*

$$d_t + \hat{V}(b_t, d_t, x_t, \nu_t) = (b + d)_t + v[Rx_t - (R - 1)(b + d)_t, \nu_t].$$

This means in an MM world with Markovian abnormal earnings, the cum-dividend valuation function $d_t + \hat{V}$ is sensitive to book value *only* via the cum-dividend combination $(b + d)_t$. Indeed, this is why b_t only appears via $(b + d)_t$ combinations on the RHS of eq. (6) in Section 1.5.

Next, introduce the pricing multiples

$$\beta(\Omega_t) \equiv \frac{\partial \hat{V}}{\partial b_t}, \; \xi(\Omega_t) \equiv \frac{\partial \hat{V}}{\partial x_t}, \; \delta(\Omega_t) \equiv \frac{\partial \hat{V}}{\partial d_t},$$

where β, ξ, and δ are the Taylor expansion coefficients in

34. This identification defines accrued and created wealth as recognized book value and abnormal earnings. Thus, it does not assume or rely on an absence of accounting bias.

$$\hat{V}(\Omega_t) = A + \beta \times (b_t' - b_t) + \xi \times (x_t' - x_t) + \delta \times (d_t' - d_t) + \cdots .$$

Like the other coefficients, dormant value $A(\Omega_t)$ may vary with Ω_t. The following theorem highlights two useful sum rules satisfied by β, ξ, and δ:

Theorem 2 (Sum Rules Theorem) *In any Markovian accounting system, MM invariance implies the book value, earnings, and dividends multiples obey the following sum rules for any* Ω_t:[35]

$$\phi \times \beta(\Omega_t) + \xi(\Omega_t) = \phi.$$
$$\phi \times \delta(\Omega_t) + \xi(\Omega_t) = 0,$$

where $\phi = R/(R - 1)$.

Proof: By Theorem 1,

$$\hat{V}(\Omega_t) = b_t + v[Rx_t - (R - 1)(b + d)_t, \nu_t].$$

Applying the chain rule to compute β and ξ yields

$$\beta = 1 - (R - 1)\frac{\partial}{\partial x_t^a} v(x_t^a, \nu_t),$$

and

$$\xi = R\frac{\partial}{\partial x_t^a} v(x_t^a, \nu_t), \ \delta = -(R - 1)\frac{\partial v}{\partial x_t^a}.$$

Linear combinations of these expressions retrieve the claimed identities. The Reader is invited to verify that the linear Ohlson solutions and the nonlinear adaptation solution obey these identities.

QED

While an analogous analysis also holds for relationships involving δ, focus now on just the relationship between β and ξ. Complementarity holds whenever pricing multiples β and ξ vary in opposite directions upon a ceteris paribus shift in earnings or book value. Specifically, V is said to exhibit earnings complementarity when

$$\text{sign}\left(\frac{\partial \beta}{\partial x_t}\right) = -\text{sign}\left(\frac{\partial \xi}{\partial x_t}\right).$$

Analogously, V exhibits book value complementarity when

$$\text{sign}\left(\frac{\partial \beta}{\partial b_t}\right) = -\text{sign}\left(\frac{\partial \xi}{\partial b_t}\right).$$

The two versions of complementarity are not mutually exclusive of each other. In fact, they mostly (although not necessarily) occur together. The following theorem reveals when complementarity is a consequence of MM invariance:

Theorem 3 *In any Markovian accounting system MM invariance implies both earnings and book value complementarity.*

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35. Note that inverting the first sum rule yields $R = \xi/(\beta + \xi - 1)$. This formula provides a potential way to estimate the implied R directly from the valuation coefficients.

Proof: *Since the linear combination $\phi\beta+\xi$ is constant by Theorem 2, β must increase whenever ξ decreases and vice versa. This immediately implies both forms of complementarity because the only requirement of complementarity is that β and ξ change in opposite directions.*

QED

Finally I establish the implications of earnings convexity. V is said to exhibit earnings convexity when

$$\frac{\partial \hat{V}}{\partial x_t} \geq 0 \text{ and } \frac{\partial^2 \hat{V}}{\partial x_t^2} \geq 0.$$

The following theorem says that in this Markovian MM world earnings convexity is equivalent to particular realizations of book value and earnings complementarity:

Theorem 4 (The Equivalence Theorem) *In any Markovian accounting system MM dividend invariance implies earnings convexity is equivalent to the following (Barth, Beaver, and Landsman style) realization of earnings and book value complementarity:*

$$\frac{\partial \beta}{\partial x_t} \leq 0 \text{ and } \frac{\partial \xi}{\partial x_t} \geq 0,$$

$$\frac{\partial \beta}{\partial b_t} \geq 0 \text{ and } \frac{\partial \xi}{\partial b_t} \leq 0.$$

Proof: Earnings convexity and the chain rule implies

$$\frac{\partial \xi}{\partial x_t} \geq 0 \text{ and } \frac{\partial \xi}{\partial b_t} \leq 0.$$

The remainder of the identities follow from earnings and book value complementarity, which is guaranteed by Theorem 3.

QED

The theorems in this section should be regarded as benchmark truisms abided by many residual income valuation functions, including the linear Ohlson and Feltham-Ohlson solutions and the contingent adaptation solution constructed herein. There is much to the Equivalence Theorem and the accounting-based formulation of MM invariance that is beyond the scope of this article. An examination of the deviations from Theorem 4 caused by different forms of MM violation will be presented elsewhere (Yee [1999]).

Appendix A.2 Proofs Related to Main Text

This appendix contains the proofs of the propositions and lemmas stated in the main text.

Proof of Lemma 1

By Eq. (11)

$$V(b_t, x_t^a) = b_t + \max_{\substack{\text{decision}\\\text{tree}}} \left[\sum_{\tau=1}^{\infty} R^{-\tau} E(x_{t+\tau}^a | b_t, x_t^a) \right]$$

where x_t^a evolves according to eq. (8). By eq. (8), $x_{t+\tau}^a$ is specified entirely by x_t^a and the firm's subsequent project choices. Moreover, the firm's decisions are based only on its existing abnormal earnings, exogenous parameters ω, R, κ, and its investment opportunity set characterized by distribution F. This means $x_{t+\tau}^a$ is *not* affected by b_t.[36] Hence

$$E(x_{t+\tau}^a | b_t, x_t^a) = E(x_{t+\tau}^a | x_t^a) \ \forall\ \tau \geq 0.$$

Since $E(x_{t+\tau}^a | x_t^a)$ is a function of x_t^a and not b_t, the most general permissible form for $V(b_t, x_t^a)$ is

$$V(b_t, x_t^a) = b_t + v(x_t^a) \tag{A.3}$$

where $v(x_t^a$ depends on all the exogenous parameters and x_t^a.

With the aid of eqs. (8) and (9), plugging this trial solution into eq. (10) yields

$$v(x_t^a) = \frac{1}{R} \max\left\{ \omega x_t^a + v(w x_t^a),\ \mu_y - \kappa + E[v(\tilde{y})] \right\}.$$

Since the second max{...} argument does not depend on $x_t^a, v(x_t^a)$ is of the form

$$v(x_t^a) = \frac{1}{R} \max\left\{ w x_t^a + v(w x_t^a),\ R v_* \right\} \tag{A.4}$$

where constant v_* equals

$$v_* \equiv \frac{1}{R}[\mu_y - \kappa + E[v(\tilde{y})]]. \tag{A.5}$$

Equation (A.4) implies $v(x_t^a) \geq v_*$ for all $x_t^a \in [\underline{y}, \bar{y}]$. Moreover, since both arguments of max{...} are nondecreasing in x_t^a, $v(x_t^a)$ must be nondecreasing in x_t^a. This suggests, assuming[37] $\underline{y} < 0$, two distinct possibilities depending on the boundary value of v in the region around[38] $x_t^a = 0$.

I. If $v(x_t^a) > v_*$ for all $x_t^a > 0$, then the v_* argument of max in eq. (A.4) is irrelevant when $x_t^a > 0$ and eq. (A.4) reduces to

$$v(x_t^a) = \frac{1}{R}[\omega x_t^a + v(\omega x_t^a)].$$

II. If there exists some $x_*^a > 0$ such that $v(x_t^a) = v_*$ for all $x_t^a \in [\underline{y}, x_*^a]$, then

$$v(x_t^a) = \begin{cases} v_* & \text{if } \underline{y} \leq x_t^a \leq x_*^a \\ R^{-1}\omega x_t^a + R^{-1} v(\omega x_t^a) & \text{if } x_*^a < x_t^a \leq \bar{y}. \end{cases}$$

---

36. A firm's decisions would depend on b_t in a model that imposes wealth constraints on project selection. By construction, the model herein does not.

37. While I consider only the case $\underline{y} < 0$, there is an analogous argument if $\underline{y} > 0$.

38. Because the iteration equation encounters a fixed point at $x_t^a = 0$, it is necessary to impose a boundary condition in the positive x_t^a region independent of what values $v(x_t^a)$ takes when $x_t^a < 0$.

where continuity[39] at x_*^e requires $Rv_* = \omega x_*^e + v(\omega x_*^e) = \omega x_*^e + v_*$ or

$$v_* = \left(\frac{\omega}{R-1}\right)x_*^e. \tag{A.6}$$

Which of these two scenarios is relevant to valuation? The trouble with any potential case I scheme is that it requires the introduction of an exogenous boundary function $h(x)$ so that for some $y_* > 0$,

$$v(x_i^e) \equiv h(x_i^e) \ \forall \ x_i^e \in [\underline{y}, y_*].$$

Such a boundary function is necessary to enable the relation $v(x_i^e) = R^{-1}\omega\, x_i^e + R^{-1}v(\omega x_i^e)$ to determine $v(x_i^e)$ unambiguously over the whole interval $[\underline{y}, \bar{y}]$. The trouble is that the adaptive firm setting provides no natural or obvious candidate for $h(x_i^e)$.

On the other hand, case II requires no such additional boundary condition. Accordingly, I shall focus exclusively on case II, which leads by induction uniquely to eq. (14). In case II,

$$v(x_i^e) = \left(\frac{\omega}{R-1}\right)x_*^e = \alpha\left(\frac{R-\omega}{R-1}\right)x_*^e \ \forall \ x_i^e \in \mathcal{S}_0,$$

where $\alpha = \omega/(R-\omega)$ and, as defined in Section 2.3, $\mathcal{S}_n$ are the subintervals between $[\underline{y}, \bar{y}]$. Hence $v(x_i^e)$ is consistent with eq. (14) when $n = 0$. Now suppose for any $n \in \{0, n_F - 1\}$ that

$$v(x_i^e) = \alpha\left\{\left(\frac{R-\omega}{R-1}\right)R^{-n}x_*^e + \left[1 - \left(\frac{\omega}{R}\right)^n\right]x_i^e\right\} \ \forall \ x_i^e \in \mathcal{S}_n.$$

Then straightforward (if tedious) algebra verifies that an equation of the same form holds for all $x_i^e \in \mathcal{S}_{n+1}$. This proves eq. (14) by induction.

Completing the proof of the Lemma requires showing the definitions stated in eqs. (12) and (13) in Section 2.3 for n_F and x_*^e are consistent with $v(x_i^e)$; n_F is just the number of intervals $\mathcal{S}_n$, which can fit into $[x_*^e, \bar{y}]$. The number n_F of such intervals is the smallest integer such that

$$\omega^{-n_F}x_*^e \geq \bar{y}.$$

Solving for n_F yields eq. (12).

As for x_*^e, combining eqs. (A.5) and (13) with the solution for $v(x_i^e)$ yields

$$\begin{aligned} x_*^e &= \left(\frac{R-1}{R\omega}\right)\{\mu_F - \kappa + E[v(\tilde{y})]\} \\ &= \left(\frac{R-1}{R\omega}\right)\left[\mu_F - \kappa + \alpha \sum_{n=0}^{n_F} \int_{y_{n-1}}^{y_n} dF(\tilde{y})\,(\gamma_n x_*^e + \lambda_n \tilde{y})\right], \end{aligned}$$

where the subinterval markers $\{y_n\}$ are as defined in Section 2.3. Simplifying the latter expression using calculus and algebraic identities such as

39. Continuity is a premise of the lemma.

$$\int_{y_{n-1}}^{y_n} dF(\tilde{y})\tilde{y} = y_n F(y_n) - y_{n-1}F(y_{n-1}) - \int_{y_{n-1}}^{y_n} d\tilde{y}\, F(\tilde{y})$$

$$= y_n F(y_n) - y_{n-1}F(y_{n-1}) - \left[\int_{\underline{y}}^{y_n} - \int_{\underline{y}}^{y_{n-1}}\right] d\tilde{y}\, F(\tilde{y}),$$

and

$$\sum_{n=0}^{n_r} \gamma_n \int_{\underline{y}}^{y_{n-1}} d\tilde{y}\, F(\tilde{y}) = \sum_{n=0}^{n_r-1} \gamma_{n+1} \int_{\underline{y}}^{y_n} d\tilde{y}\, F(\tilde{y})$$

yields, with the aid of eq. (15) of Section 2.3 and eq. (A.9) of Appendix A.3,

$$x_r^e = \frac{R-1}{R\omega}\left\{\mu_r - \kappa + \alpha\gamma_{n_r} x_r^e + \alpha\lambda_{n_r}\mu_r + \sum_{n=0}^{n_r-1}\left(\frac{\omega}{R}\right)^{n+1}\int_{\underline{y}}^{\omega^{-n}x_r^e} d\tilde{y}\, F(\tilde{y})\right\}.$$

Plugging in the values for γn_r and λ_{n_r} and algebraically simplifying yields eq. (13).

Finally, $V(b_r, x_r^e)$ is convex in x_r^e because (1) $V(b_r, x_r^e)$ is continuous and (2) piecewise linear in x_r^e and (3) the slopes λ_n of each linear piece of $V(b_r, x_r^e)$ are nondecreasing in x_r^e. It is a rudimentary mathematical property that any function satisfying properties 1 to 3 is convex.

QED

Derivation of Figure 7 Entries

The relationship between $V^{\text{non-adaptive}}(b_r, x_r^e) = b_r + \alpha x_r^e$ and the various parameters is obvious. Moreover, derivation of the ω-, R-, and Δ_r-related entries are indicated in the text of Section 2.6.

All that remains is the κ and μ_r column of entries. For κ the analogous argument as that given for Δ_r goes through quite easily. However, it is instructive to consider an alternative argument, which holds whether n_r is finite or infinite. From above

$$x_r^e = \left(\frac{R-1}{R\omega}\right)[\mu_r - \kappa + E[\upsilon(\tilde{y})]], \tag{A.7}$$

where $\tilde{y} \in F$ and, if $\tilde{y} \in S_n$,

$$\upsilon(\tilde{y}) = \alpha\left\{\left(\frac{R-\omega}{R-1}\right)R^{-n}x_r^e + \left[1 - \left(\frac{\omega}{R}\right)^n\right]\tilde{y}\right\},$$

where $\upsilon(\tilde{y})$ depends not only on $\tilde{y}$, but also on[40] ω, R, and x_r^e:

$$\upsilon(\tilde{y}) \equiv \upsilon(\tilde{y}; \omega, R, x_r^e),$$

where by eq. (13), $x_r^e = x_r^e(\omega, R, \kappa, F)$. Examining the partial derivative of υ with respect to x_r^e yields

$$0 \le \frac{\partial \upsilon}{\partial x_r^e}\bigg|_{\tilde{y},\omega,R} \le \frac{\omega}{R-1}.$$

40. Any dependence of $\upsilon(\tilde{y})$ on n_r can be reduced away by replacing n_r with its expression, eq. (12), in terms of ω and x_r^e. Note that κ and F do not enter into the expression for $\upsilon(\tilde{y})$ explicitly; $\upsilon(\tilde{y})$ depends on κ and F only indirectly through x_r^e.

In turn, this implies the useful bound

$$0 \le \frac{\partial \mathcal{G}}{\partial x^*_\infty}\bigg|_{\omega,R,F} \le \frac{\omega}{R-1},$$

where $\mathcal{G}(\omega, R, x^*_\infty, F) \equiv E[v(\tilde{y})]$.

Applying the Implicit Function Theorem to eq. (A.7) yields

$$\left\{1 - \left(\frac{R-1}{\omega R}\right)\frac{\partial \mathcal{G}}{\partial x^*_\infty}\bigg|_{\omega,R,F}\right\}\frac{\partial x^*_\infty}{\partial \kappa}\bigg|_{\omega,R,F} = -\frac{R-1}{\omega R} < 0.$$

Since

$$\frac{\partial \mathcal{G}}{\partial x^*_\infty}\bigg|_{F,\omega,R} \le \frac{\omega}{R-1},$$

this means the left-hand side (LHS) quantity in brackets is positive. Hence, negativity of the RHS implies $\partial x^*_\infty/\partial\kappa_{\omega,R,F} < 0$. This inequality and the relationships of n_p, IOA(0), and $E(\tau)$ to x^*_∞, leads easily to the first column of Figure 7.

Unfortunately, because $\mathcal{G}$ depends explicitly on ω, R, and F the same approach does not work for determining the response of x^*_∞ to perturbations of these parameters. This is why I have not been able to complete the comparative statics analysis of ω and R except in the special case when $\Delta_F = 0$.

The derivation of x^*_∞'s μ_F dependence is subtle. Taking the partial derivative with respect to μ_F of eq. (17) and rearranging yields

$$\left[1 - \left(\frac{R-1}{R^2}\right)\sum_{n=0}^{\infty} R^{-n}F(\omega^{-n}x^*_\infty)\right]\frac{\partial x^*_\infty}{\partial \mu_F}\bigg|_{\omega,R,\kappa,\Delta_F}$$
$$= \frac{R-1}{(R-\omega)\omega} + \frac{R-1}{R^2}\sum_{n=0}^{\infty}\left(\frac{\omega}{R}\right)^n \frac{\partial}{\partial \mu_F}\bigg|_{x^*_\infty,\omega,R,\kappa,\Delta_F}\int_{\underline{y}}^{\omega^{-n}x^*_\infty} d\tilde{y}\, F(\tilde{y}). \tag{A.8}$$

The bracketed factor on the LHS is strictly positive as explained in Section 2.6. Regarding the RHS, eq. (A.9) below implies

$$\mu_F = \bar{y} - \int_{\underline{y}}^{\omega^{-n}x^*_\infty} d\tilde{y}\, F(\tilde{y}) - \int_{\omega^{-n}x^*_\infty}^{\bar{y}} d\tilde{y}\, F(\tilde{y}).$$

Taking the partial derivative with respect to μ_F of both sides implies

$$\frac{\partial}{\partial \mu_F}\bigg|_{x^*_\infty,\omega,R,\kappa,\Delta_F}\int_{\underline{y}}^{\omega^{-n}x^*_\infty} d\tilde{y}\, F(\tilde{y}) = -1 - \frac{\partial}{\partial \mu_F}\bigg|_{x^*_\infty,\omega,R,\kappa,\Delta_F}\int_{\omega^{-n}x^*_\infty}^{\bar{y}} d\tilde{y}\, F(\tilde{y}).$$

Since increasing μ_F with all else fixed shifts the smaller $[F(\tilde{y}) < 1]$ part of $F(\tilde{y})$ toward $\bar{y}$, it lowers the value of $\int_{\omega^{-n}x^*_\infty}^{\bar{y}} d\tilde{y}F(\tilde{y})$. This means

$$\frac{\partial}{\partial \mu_F}\bigg|_{x^*_\infty,\omega,R,\kappa,\Delta_F}\int_{\omega^{-n}x^*_\infty}^{\bar{y}} d\tilde{y}\, F(\tilde{y}) \le 0.$$

Together with the previous expression, this implies

$$\frac{\partial}{\partial \mu_{y^t}}\bigg|_{\substack{x_t^*, \omega, R, \\ \kappa, \Delta_y}} \int_{\underline{y}}^{\omega^{-n} x_t^*} d\tilde{y}\, F(\tilde{y}) \geq -1$$

so that

$$\frac{R-1}{R^2}\sum_{n=0}^{\infty}\left(\frac{\omega}{R}\right)^n \frac{\partial}{\partial \mu_{y^t}}\bigg|_{\substack{x_t^*, \omega, R, \\ \kappa, \Delta_y}} \int_{\underline{y}}^{\omega^{-n} x_t^*} d\tilde{y}\, F(\tilde{y}) \geq \frac{R-1}{R^2}\sum_{n=0}^{\infty}\left(\frac{\omega}{R}\right)^n (-1)$$

$$= -\left[\frac{R-1}{(R-\omega)R}\right].$$

Therefore a lower bound on the RHS of eq. (A.8) is

$$\left(\frac{R-1}{R-\omega}\right)\left(\frac{1}{\omega}-\frac{1}{R}\right) > 0.$$

Since $R > \omega$, one concludes that

$$\frac{\partial x_t^*}{\partial \mu_{y^t}}\bigg|_{\substack{\omega, R, \\ \kappa, \Delta_y}} > 0.$$

QED

Appendix A.3 Appendix Lemmas

This subsection describes a useful lemma pertaining to the quality distribution F of project opportunities.

Appendix Lemma 1 (*adapted from Rothschild and Stiglitz* [1970], [1971]) *Consider any two cumulative distribution functions* F_1 *and* F_2 *which* (*a*) *have support only in, say,* $[\underline{y}, \bar{y}]$; (*b*) *have identical means; and* (*c*) *satisfy a single-crossing condition* [*that is,* $F_2(y) = F_1(y)$ *only at one point* $y \in (\underline{y}, \bar{y})$]. *Then* F_2 *has greater spread than* F_1 *iff*

$$\int_{\underline{y}}^{y} dy\,[F_2(y) - F_1(y)] \geq 0 \ \forall\ y \in [\underline{y}, \bar{y}].$$

In other words, the integral

$$\int_{\underline{y}}^{y} dy\, F(y)$$

is a monotonic indicator of the spread of c.d.f. function F.

Proof: Assumption (a) implies $F(\underline{y}) = 0$ and $F(\bar{y}) = 1$. Integration by parts implies the mean corresponding to a distribution may be expressed as

$$\mu \equiv \int_{\underline{y}}^{\bar{y}} dF\, y = \bar{y} - \int_{\underline{y}}^{\bar{y}} dy\, F(y). \tag{A.9}$$

Hence assumption (b) implies

$$\int_{\underline{y}}^{\bar{y}} dy\, [F_2(y) - F_1(y)] = 0. \tag{A.10}$$

Suppose F_1 and F_2 satisfy assumption (c) and, without loss of generality, F_2 has greater spread than F_1. By definition of "spread," this means

$$F_2(y) \begin{cases} > F_1(y) & \text{if } y \in (\underline{y}, \hat{y}); \\ = F_1(y) & \text{if } y = \hat{y}; \\ < F_1(y) & \text{if } \underline{y} \in (\hat{y}, \bar{y}). \end{cases}$$

This implies that

$$\int_{\underline{y}}^{y} dy\, [F_2(y) - F_1(y)] > 0\ \forall\ y \in (\underline{y}, \hat{y}).$$

All that remains is to prove

$$\int_{\underline{y}}^{y} dy\, [F_2(y) - F_1(y)] \geq 0\ \forall\ y \in (\hat{y}, \underline{y}].$$

Since this inequality is identically true when $y = \underline{y}$ by eq. (A.10), it is necessary to focus on just $(\hat{y}, \underline{y})$. Suppose there exists some $y_* \in (\hat{y}, \underline{y})$ such that

$$\int_{\underline{y}}^{y*} dy\, [F_2(y) - F_1(y)] < 0.$$

But since $F_2(y) < F_1(y)$ for all $y \in (\hat{y}, \underline{y})$, this would require

$$\int_{\underline{y}}^{\bar{y}} dy\, [F_2(y) - F_1(y)] < 0,$$

which contradicts the identity eq. (A.10).

QED

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