# Central charge of the parallelogram-lattice strong coupling Schwinger model

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We put forth a Fierz-transformed hopping expansion for strong coupling Wilson fermions. As an application, we show that the strong coupling Schwinger model on parallelogram lattices with nonbacktracking Wilson fermions span, as a function of the lattice skewness angle, the $\Delta = -1$ critical line of six-vertex models. This Fierz-transformed formulation also applies to backtracking Wilson fermions, which as we describe apparently correspond to richer systems. However, we have not been able to identify them with exactly solved models.

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## I. INTRODUCTION AND RESULTS

In recent years there has been remarkable progress in the classification of two-dimensional spin systems and three-dimensional topological Yang-Mills theories in relation to conformal field theories [1]. Yet, despite considerable activity in lattice gauge theories with fermions [2] and two dimensional lattice toy models [3], the Wilson lattice transcription of the Schwinger model [4] remains unsolved. The only exactly solved model with Wilson fermions and local gauge invariance is the square lattice strong coupling Schwinger model, whose partition function $Z_\square$ at infinite hopping constant equals $Z_{6V}[\frac{1}{2}, \frac{1}{2}, 1]$, a 6-vertex model partition function [5]. By known 6-vertex model features, this mapping reveals that $Z_\square$ is critical and its continuum limit is a conformal field theory with central charge $c = 1$.

Since the 6-vertex model has a nontrivial phase structure, it is natural to seek the $Z_\square$ extensions which map to other regions of the $Z_{6V}[A, B, C]$ parameter space. To this end, we put forth a Fierz-transformed strong coupling hopping expansion for actions of the form

$$S_F = \sum_{x\in\Lambda}\Big(-M\overline{\psi}_x\psi_x + \sum_{\mu=0}^{1}(\,\overline{\psi}_{x+\hat{\mu}}T^{(+,\mu)}U^{\dagger}_{x,\mu}\psi_x + \overline{\psi}_x T^{(-,\mu)}U_{x,\mu}\psi_{x+\hat{\mu}})\Big), \tag{1a}$$

$$\Lambda \equiv \left\{\sum_{\mu=0}^{1} x^\mu \hat{e}_\mu | x^\mu \in Z\right\}, \tag{1b}$$
$$M \equiv m + \sum_{\mu=0}^{1}(T^{(+,\mu)} + T^{(-,\mu)}).$$

$S_F$ is the Wilson fermion action on a square lattice action if $\{\hat{e}_1, \hat{e}_2\}$ are orthonormal and $T^{(\pm,\mu)} = \frac{1}{2}(r \pm \gamma^\mu)$. More generally, on a parallelogram lattice defined by

$$\hat{e}_\mu \equiv \begin{pmatrix}\hat{e}^{(0)}_\mu \\ \hat{e}^{(1)}_\mu\end{pmatrix}, \quad \hat{e}_0 \equiv \lambda\begin{pmatrix}1\\0\end{pmatrix}, \quad \hat{e}_1 \equiv \lambda'\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \tag{2a}$$

$$g \equiv \sum_{a,b=0}^{1} \hat{e}^{(a)}\hat{e}^{(b)}\delta_{ab} = \begin{pmatrix}\lambda^2 & \lambda\lambda'\cos\theta \\ \lambda\lambda'\cos\theta & \lambda'^2\end{pmatrix}, \tag{2b}$$

the strong coupling partition function is

$$Z_{\rm SC} \equiv \int_F \int \prod_{x,\mu\in\Lambda} dU_{x,\mu}\, e^{-S_F}\ , \tag{3a}$$

$$\int_{F_x} \equiv \int d\psi^1_x d\overline{\psi}^1_x d\psi^2_x d\overline{\psi}^2_x, \qquad \int_F \equiv \prod_{x\in\Lambda}\int_{F_x}\ . \tag{3b}$$

Assuming the nonbacktracking condition

$$T^{(-,\mu)}T^{(+,\mu)} = 0 = T^{(+,\mu)}T^{(-,\mu)}, \tag{4}$$

which prevents hopping expansion quarks from backtracking, we will show that

$$Z_{\rm SC} = \int_F e^{-S_{\rm SC\mathit{F}}}, \qquad S_{\rm SC\mathit{F}} = \sum_{x\in\Lambda}\Big(-M\overline{\psi}_x\psi_x + \sum_{\mu=0}^{1}\Theta^{(+,\mu5)}_x\Theta^{(-,\mu5)}_{x+\hat{\mu}}\Big). \tag{5}$$

Commuting operators $\Theta^{(\epsilon,\mu5)}_x$ are characterized by[1]

[1]Under the Euclidean analogue of Hermitian conjugation, $\psi^\dagger_E \equiv \overline{\psi}\gamma_5,\ \overline{\psi}^\dagger_E = \gamma_5\psi,\ (S_F)^\dagger_E = S_F,\ (\Theta^{(\epsilon,\mu5)}_x)^\dagger_E = \Theta^{(\epsilon,\mu5)}_x$.

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$$\int_{F_x} (\overline{\psi}_x \psi_x)^q \left(\Theta_x^{(\epsilon,\mu 5)}\right)^p = 2\delta_{q,2}\ \delta_{p,0} \quad (\forall\ q,p \in \{0,1,2,\cdots\}), \tag{6a}$$

$$-\int_{F_x} \Theta_x^{(\epsilon',\nu 5)}\Theta_x^{(\epsilon,\mu 5)} = r^{(\mu)}r^{(\nu)}\left(\delta_{\mu\nu}\delta_{-\epsilon'\epsilon} + \delta_{\mu 0}\delta_{\nu 1}S^2_{-\epsilon',\epsilon} + \delta_{\mu 1}\delta_{\nu 0}S^2_{\epsilon,-\epsilon'}\right), \tag{6b}$$

where $r^{(\mu)}$ and $S_{\epsilon,\epsilon'}$ are functions (given below) of $\lambda$, $\lambda'$ and $\theta$. Since $\Theta_x^{(+,\mu 5)}\Theta_x^{(-,\nu 5)}$ completely saturates $\int_{F_x}$, $\Theta_x^{(\epsilon,\mu 5)}$ hops along self-avoiding paths. Monomers $(M\overline{\psi}_x\psi_x)^2$ fill in unhopped sites. We call this the Fierz-transformed hopping expansion because the "$\Theta_x^{(+,\mu 5)}\Theta_{x+\hat{\mu}}^{(-,\mu 5)}$" form in (5) is achieved using Fierz identities. As described in Sec. III, the Fierz-transformed hopping expansion also applies to backtracking models.

Following (5), $\Theta_x^{(\epsilon,\mu 5)}$ hopping amplitudes are identified in Sec. II with Boltzmann weights of two-state vertex models. A nonperturbative solution of these vertex models (by Bethe ansatz or whatever) is tantamount to resummation of the hopping expansion. In this way we identify $Z_{\rm SC}$ with $Z_{8V}$, the 8-vertex model partition function with Boltzmann weights:

$$\omega_1 = M^2, \quad \omega_2 = 0, \tag{7a}$$

$$\omega_3 = (\lambda'\sin\theta)^{-2}, \quad \omega_4 = (\lambda\sin\theta)^{-2}, \tag{7b}$$

$$\omega_5 = \omega_6 = \frac{1}{4\lambda\lambda'}\left(\frac{1}{\cos\frac{\theta}{2}}\right)^2, \quad \omega_7 = \omega_8 = \frac{1}{4\lambda\lambda'}\left(\frac{1}{\sin\frac{\theta}{2}}\right)^2. \tag{7c}$$

The $\{\omega_i\}$ are defined in, for example, p. 347 of Lieb and Wu [6]: $\omega_1$ corresponds to the empty vertex, $\omega_3$ and $\omega_4$ to vertical and horizontal lines, $\omega_5$ and $\omega_6$ to lower-right and upper-left corners, and $\omega_7$ and $\omega_8$ to lower-left and upper-right corners. Setting $\theta = \pi/2$ and $r^{(\mu)} = 1$ recovers the square lattice result of Ref. [5].

The 8-vertex model is not solved in general for the Boltzmann weights given in (7a)–(7c). A solvable case is $M = 0$ and $\lambda' = \lambda$. In this subspace, define

$$D \equiv \omega_1 = \omega_2 = 0, \quad C \equiv \omega_3 = \omega_4, \qquad A \equiv \omega_5 = \omega_6, \quad B \equiv \omega_7 = \omega_8. \tag{8}$$

Then 8-vertex model symmetries [6] imply

$$Z_{\rm SC} = Z_{8V}[D,C,A,B] = Z_{8V}[A,B,C,D] = Z_{6V}[A,B,C]\ . \tag{9}$$

In the thermodynamic limit, these 6-vertex models fall on the critical line characterized by Lieb parameter [6]

$$\Delta \equiv \frac{A^2+B^2-C^2}{2AB} = -1. \tag{10}$$

Varying the skewness angle $\theta$ from 0 to $\pi$ spans the critical line between the antiferroelectric and disorder phases of the 6-vertex model.

While the points on this critical line all have a central charge $c = 1$ [7], to our knowledge it has not been demonstrated whether they are the same $c = 1$ conformal field theory or different ones. Renormalization group arguments suggest that the subset of points $A = B = \frac{C}{2}$, the so-called $F$ model, corresponding to $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ can be identified with free massless boson field theory [8].

## II. FIERZ-TRANSFORMED STRONG COUPLING HOPPING EXPANSION

In the finite coupling ($\beta > 0$) hopping expansion [2] $\psi$ hops from site to site with weight $T^{(\epsilon,m)}U_{x,m}$. At strong coupling such quark motions are suppressed. Define

$$D^{(\mu)}_{il,jk} \equiv T^{(-,\mu)}_{ij}\ T^{(+,\mu)}_{kl}, \quad \Xi(x,y;D) \equiv \overline{\psi}^i_x\psi^l_x D_{il,jk}\psi^j_y\overline{\psi}^k_y. \tag{11}$$

Since $(\Xi)^3 = 0$ in two dimensions, integrating out U(1) links $\{U_{x,\mu}\}$ using

$$\int dU \exp(aU^\dagger + bU) = \sum_{k=0}^{\infty}(ab)^k/(k!)^2$$

yields

$$Z_{\rm SC} = \int_F \exp(M\sum_x \overline{\psi}_x\psi_x)\prod_{x,\mu}[1 + \Xi(x,x+\hat{\mu};D^{(\mu)}) + \frac{1}{4}\Xi^2(x,x+\hat{\mu};D^{(\mu)})] \tag{12a}$$

$$= \int_F \exp\left(\sum_x [M\overline{\psi}_x\psi_x + \sum_\mu(\Xi(x,x+\hat{\mu};D^{(\mu)}) - \tfrac{1}{4}\Xi^2(x,x+\hat{\mu};D^{(\mu)}))]\right). \tag{12b}$$

Let $n,m \in \{\pm 0, \pm 1\}$, $T^{(\epsilon,-m)} \equiv T^{(-\epsilon,m)}$, and $k_{y,-n} \equiv k_{y-\hat{n},n}$. Expanding $\exp(M\overline{\psi}_x\psi_x) = \sum_{s_x=0}^{2}(M\overline{\psi}_x\psi_x)^{s_x}/s_x!$ reveals that $\psi\otimes\overline{\psi}$ hops with amplitude $D^{(m)}$ such that $s_x + \sum_{m=-1}^{1}k_{x,m} = 2$ for each $x \in \Lambda$. Thus only three events are possible at each site: (13a)$s_x = 2$, $k_{x,m} = 0$; (13b)$s_x = 1$, $k_{x,m} = 1$; or (13c)$s_x = 0$, $k_{x-\hat{n},n} = k_{x,m} = 1$. The associated amplitudes are

$$\int_{F_x} \frac{1}{2}(M\overline{\psi}_x\psi_x)^2 = M^2\ , \tag{13a}$$

$$\int_{F_x} M\overline{\psi}_x\psi_x\ \Xi(x,x+\hat{m};D^{(m)}) = M{\rm tr}(T^{(+,m)}T^{(-,m)}\psi_{x+\hat{m}}\overline{\psi}_{x+\hat{m}}), \tag{13b}$$

$$\int_{F_x} \Xi(x-\hat{n},x;D^{(n)})\ \Xi(x,x+\hat{m};D^{(m)}) = \Xi(x-\hat{n},x+\hat{m};\widetilde{D}^{(n,m)}), \tag{13c}$$

$$\widetilde{D}^{(n,m)}_{il,jk} \equiv (T^{(-,n)}T^{(-,m)})_{ij}(T^{(+,m)}T^{(+,n)})_{kl} - (T^{(-,n)}T^{(+,n)})_{il}(T^{(+,m)}T^{(-,m)})_{kj}. \tag{13d}$$

A backtracking $\psi \otimes \overline{\psi}$ pair, $k_{x,m} = 2$ or $n = -m$ in (13c), saturates $\int_F$ at the two sites it occupies and, hence, makes a dimer. Since 2-state vertex models cannot model dimer-loop mixtures, $Z_{\rm SC}$ is not a 2-state vertex model unless backtracking is forbidden. In this section we adopt (4), which sets $\widetilde{D}^{(n,-n)} = 0$. Then $\psi \otimes \overline{\psi}$ world lines comprise a self-avoiding loop gas with Boltzmann weight $M^2$ for unoccupied sites and weights $\widetilde{D}^{(n,m)} = T^{(-,n)}T^{(-,m)} \otimes T^{(+,m)}T^{(+,n)}$ along loops.

We now recast the problem so that the Boltzmann weights are more succinctly related to hopping amplitudes. Define $\gamma_5 \equiv i\gamma_0\gamma_1$,

$$\gamma_0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^\mu = \sum_{\nu,a} g^{\mu\nu}\hat{e}^{(a)}_\nu \gamma_a, \tag{14a}$$

$$T^{(\pm,\mu)} \equiv \frac{1}{2}(r^{(\mu)} \pm \Gamma^\mu), \tag{14b}$$

$$r^{(0)} = (\lambda \sin\theta)^{-1}, \quad r^{(1)} = (\lambda' \sin\theta)^{-1}.$$

Wilson regulators $\{r^{(\mu)}\}$ are chosen so that, in addition to (4),

$$T^{(\epsilon,\mu)}_{ij}T^{(-\epsilon,\mu)}_{kl} = (T^{(\epsilon,\mu)}\gamma_5)_{il}\ (T^{(-\epsilon,\mu)}\gamma_5)_{kj} \quad (\mu \text{ fixed}), \tag{14c}$$

$$T^{(\epsilon,\mu)}T^{(\epsilon',\mu)} = \delta^{\epsilon\epsilon'}\ r^{(\mu)}T^{(\epsilon,\mu)}, \tag{14d}$$

$$r^{(1)}\ T^{(\epsilon,0)} = r^{(0)}\ S^\dagger\ T^{(\epsilon,1)}\ S\ ,$$

$$S = \begin{pmatrix} \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ -\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix} \equiv \begin{pmatrix} S_{+,+} & S_{+,-} \\ S_{-,+} & S_{-,-} \end{pmatrix}. \tag{14e}$$

Fierz identity (14c) implies

$$\Xi(x,x+\hat{m},D^{(m)}) = -\Theta^{(-,m5)}_x\Theta^{(+,m5)}_{x+\hat{m}}, \tag{15}$$

$$\Theta^{(\epsilon,\mu5)}_x \equiv \overline{\psi}_x T^{(\epsilon,\mu)}\gamma_5\psi_x$$

and transforms (12b) to (5). Following (5), $\Theta^{(\epsilon,\mu5)}_x$ hops from direction $n \equiv \text{sgn}(n)\nu$ to direction $m \equiv \text{sgn}(m)\mu$ with amplitude (6b) where $\epsilon' = \text{sgn}(n)$ and $\epsilon = -\text{sgn}(m)$. Boltzmann weights $\{\omega_i\}$ can be read off from (6b). Straight $n = m$ vertical or horizontal hops, corresponding to $\omega_3$ and $\omega_4$, have weight $(r^{(n)})^2$. If $x$ is approached from $n = 0$ and exited in the $m = 1$ direction, $\omega_6 = r^{(0)}r^{(1)}S^2_{-,-}$. Similarly, $n = -1$ and $m = 0$ yield $\omega_7 = r^{(0)}r^{(1)}S^2_{+,-}$. Equations (7a)–(7c) list the remaining vertices.

## III. BACKTRACKING

To study backtracking effects when (4) is removed, we work on a square lattice with $T^{(\pm,\mu)} \equiv \frac{1}{2}(r \pm \gamma^\mu)$. Fixed-$\mu$ Fierz transformations are

$$T^{(\epsilon,\mu)}_{ij}T^{(-\epsilon,\mu)}_{kl} = (T^{(\epsilon,\mu)}\gamma_5)_{il}(T^{(-\epsilon,\mu)}\gamma_5)_{kj} + \Big(\frac{r^2-1}{8}\Big)\sum_\sigma (-1)^\sigma \gamma^\sigma_{il}\gamma^\sigma_{kj}, \tag{16}$$

where $\gamma^\sigma \in \{\gamma^s \equiv 1, \gamma_5, \gamma_0, \gamma_1\}$, $(-1)^\sigma = -1$ for $\gamma_5$, and $(-1)^\sigma = +1$ otherwise. Hermitian operators

$$\Theta^\sigma_x \equiv \begin{cases} \overline{\psi}_x\gamma^\sigma\psi_x & \text{if } \gamma^\sigma \in \{\gamma^s, \gamma_5, T^{(\epsilon,\mu)}\gamma_5\}, \\ i\overline{\psi}_x\gamma^\sigma\psi_x & \text{if } \gamma^\sigma \in \{\gamma_0, \gamma_1\}, \end{cases} \tag{17}$$

are characterized by

$$\int_{F_x} (\Theta^s_x)^2 = 2, \qquad (\Theta^s_x)^3 = 0, \tag{18a}$$

$$\Theta^{\sigma'}_x\ \Theta^\sigma_x = (-1)^\sigma\ \delta_{\sigma'\sigma}\ (\Theta^s_x)^2 \qquad (\sigma,\sigma' \in \{s,5,0,1\}), \tag{18b}$$

$$\Theta^\sigma_x\ \Theta^{(\epsilon,\mu5)}_x = \Big[\frac{1}{4}(\epsilon\epsilon'\delta_{\mu\nu} - r^2)\ \delta_{\sigma(\epsilon',\nu5)} - \frac{r}{2}\ \delta_{\sigma5} + \frac{\epsilon}{2}\ \epsilon^{\mu\sigma}\Big]\ (\Theta^s_x)^2. \tag{18c}$$

Following (12b) and (16),

$$S_{SCF} = S^s_{SCF} + \sum_{x,\mu\in\Lambda}\Big[\Theta^{(-,\mu5)}_x\Theta^{(+,\mu5)}_{x+\hat{\mu}} + \Big(\frac{1-r^2}{8}\Big)\sum_{\sigma=5,0,1}\Theta^\sigma_x\Theta^\sigma_{x+\hat{\mu}}\Big], \tag{19a}$$

$$S^s_{SCF} \equiv \sum_{x\in\Lambda}\Big[-M\Theta^s_x + \frac{1}{32}\sum_{\mu=0}^{1}\Big((r^2-1)\Theta^s_x\Theta^s_{x+\hat{\mu}}+2\Big)^2\Big]. \tag{19b}$$

Note that $S^s_{SCF}$ contains dimers. These systems are currently under study. We have not been able to identify them with solved models.

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